ANNEX

COLLEGE MATHEMATICS DURING RECONSTRUCTION*

R. W. BRINK, University of Minnesota

1. Introduction. If we pause for a little while amid the turmoil of the war-training courses in order to look forward to the period just after the war, it is not because we feel that the war is over nor even that its end is hidden from us by being just around the corner. The war-training programs will doubtless be with us for a long time to come. Doubtless they will be subject to more of the sudden and arbitrary changes that we have all found so baffling. Doubtless we shall continue to be faced with shortages of staff, irregular schedules, budgetary difficulties, and elusive academic standards, as in the recent past. Yet I feel that the general pattern of training in the colleges is fairly well established or at least that we are psychologically better prepared than we were twelve months ago to meet such wartime changes as may occur. On this plateau, then, rather than summit, that we have reached I should like to pause to take stock of our gains and losses, and to look forward toward the land ahead. Perhaps it is not too early to see something of this land of college mathematics after the war is won and of the problems we shall find there. Perhaps, by looking back at the way we have come, we can find help in solving those problems.

2. College enrollments after the war. In order to form some notion of the size of the task ahead of us, let us first consider the probable general university and collegiate enrollment, without special regard to its effect on mathematics. We shall almost certainly have a flood of former students returning to the colleges from their service with the armed forces, governmental agencies, and defense plants. To these will be added the recent high school graduates of perhaps several years, who will be entering college for the first time.

Another force, also tending in the direction of increased enrollments, and, to my mind, at least equally important for the long haul, is the probable broader base of education that will result from the war. American education, in constrast to that in most European countries, has long been characterized by the theory of the broad base. Not for us has been the system of higher education only for the intellectually or economically elite. Of course, this has resulted in the attempted education of many young persons never intended by their Maker to breathe the rare atmosphere of the intellectual life. Yet if, as educators would probably be the first to assert, education is a chief bulwark of democracy, it would seem to be a sound theory which both permits and encourages each citizen to carry his education as far as his native gifts permit, regardless of his economic status and arbitrary academic standards. In the experiences that our young men are undergoing in the training camps and on the battlefields of the world, I can see a great levelling force at work. It will level upward, I hope, as well as downward. After observing that some eighty per cent of their officers

* Retiring presidential address delivered to the Mathematical Association of America, Chicago, Ill., November 27, 1943.

have had some college background, after having taken part in Army Institute and other courses preparing them for their immediate jobs or more remotely for their places as citizens, after having worked and fought on even terms with comrades who expect to return to college, is it not likely that many thousands of our young soldiers will demand for themselves the privileges of further education?

When they knock at our college doors, I hope that we shall apply some intelligent guidance, not so much in the form of outright rejection as in advice as to the most suitable types of training. But somewhere in our educational system we shall accept them without too rigid an application of scholastic requirements. For one thing, during our recent experiences, we have learned not to be too choosey about our raw material. Many colleges that previously accepted only students in the upper twenty per cent of their high school classes have been delighted to take men selected for them on the doubtful basis of an Army General Classification Test. Good men are still good men, and stand out as such. But we have found that with the aid of proper refresher courses even men of inferior preparation can be given useful training at a level beyond what might at first be expected from their previous experience. I shall revert to this point later on.

Most of all we will give an ungrudging welcome to these young men and women because we think that it will be advantageous to the nation to do so. Much as we may fear future wars and foreign aggression, our greatest dangers after the war will come from within. They will come from the disunity of one special interest opposing others. To dispel such dissensions we must teach as large a part of your young population as possible to distinguish information from dishonest propaganda, to seek logical instead of emotional bases of action, to understand the essentially cooperative nature of society, to respect he other man's manner of thought, and to earn a livelihood in socially useful work.

Not all of the tendencies will be in the direction of increased enrollments. Many of the returning men will be impatient to secure permanent positions. Many will have new opportunities open to them as a result of their war training and experience. Others, who married on the strength of wartime salaries and find that they married on a shoestring, will be forced to seek work to support their families.

But, on the whole, the signs point to enrollments far exceeding anything that the colleges have known before. A committee at one of our large middle-western state universities* has estimated that during the first year after the war its enrollment will be from eleven per cent to seventeen per cent, depending on the war's duration, greater than it ever was before, and that during the second year after the war it will be from twenty-eight per cent to thirty-four per cent greater. By 1950–1951, the committee predicts an enrollment forty per cent greater than the maximum hitherto. These estimates do not take into ac-

* Report of the Senate Committee on Education, University of Minnesota, Senate Minutes, October 21, 1943.

count the stimulus of any possible plans for federal subsidization of students after the war. If these plans develop, it is likely that the forty per cent increase may occur within a year or two after the war's end.

With reference to the share that returning service men will occupy in our load, Colonel John N. Andrews of the Reemployment Division of Selective Service Headquarters states that the general indication now is that of the eleven to twelve million men who will have been in service by the time the war ends, at least one million will wish to return to vocational schools or colleges and universities to pursue further study. Many of these entered the service directly from high schools. Large numbers will have completed from one to three years of college work, and will wish to continue in their original courses or in other directions. And the indications are that those who desire to return to school will be given financial support to enable them to do so.

3. Plans for federal subsidization of education. Without yet knowing the exact form that a scheme of federal subsidies will assume, we can take it as probable that some scheme will develop. The most definite suggestions from any authoritative source are contained in the recommendations of the Conference on Postwar Readjustment of Civilian and Military Personnel under the auspices of the National Resources Planning Board.* The general principles of these recommendations were contained in the report of a special committee on education appointed by the President of the United States. On October 27, 1943, the President embodied these principles in a message† to Congress with a request for the necessary legislation. In brief outline, the recommendations for subsidy of education call for two plans. Under the first plan of general education free tuition and reasonable allowances for maintenance would be provided for a period not exceeding one year to any honorably discharged ex-service man who might apply for it. Both general and vocational or professional education and training would be provided, but the latter would not be provided in those fields and for those occupations in which the supply of trained personnel is already sufficient to meet anticipated requirements. Training and education would begin at the level appropriate to the individual and should have such forms and methods as are suited to the needs of mature men regardless of the academic level. (Recommendations 50–58.) In addition to this general plan of education the Conference recommended a plan of supplementary education. This plan recognizes the special responsibility of the Nation to those who had entered upon an extended course of education which was interrupted by their military service and it also recognizes the need of the Nation for specially trained personnel. Under the program of supplementary education scholarships not to exceed four years in duration would be granted on a competitive basis in the fields of higher education and in such technical and professional fields as

* Demobilization and Readjustment—Report of the Conference on Postwar Readjustment of Civilian and Military Personnel, National Resources Planning Board, June, 1943, Superintendent of Documents, Washington, D. C.

† Congressional Record, October 27, 1943, p. 8881.

offer some likelihood of satisfactory and useful employment. (Recommendations 59–66.)

We cannot be certain that such a program of federal subsidization will be adopted. It seems very probable, however, that some very similar plan will take effect. For the purpose of discussion, I should like to assume that this is the case. And I should like to consider some aspects of the situation in college mathematics with such a program imposed on our other tasks.

4. The load in mathematics. What the Conference called the "general" plan of education provides for one year of either general education or training for vocations or professions in which there is a predictable opportunity for employment. Even before a comprehensive survey of the needs in the various trades and professions has been made, we can make some prediction of the impact of this program on the mathematical load. Certainly, if we recall the importance that has been attached by the armed forces and by industry to elementary training in mathematics for their mechanics and foremen, we cannot doubt that there will be a great deal to do in teaching what we think of as "shop mathematics." I hope that most of this teaching can be done in trade schools, some special junior colleges, and in special extension courses of the universities or in industry itself. For I am sure that collegiate staffs will be fully occupied at more nearly the collegiate level.

But what of the young men who in ordinary times would have gone directly from high school into jobs requiring no further academic training? Many thousands of them who will have received training and seen service as electricians' mates or airplane mechanics or radio technicians will be stirred to seek advancement in similar lines. Many of them will wish to spend the time of their subsidized training as students of engineering or science. To the extent to which they can meet or nearly meet the entrance requirements of our colleges, they can be absorbed into the regular engineering and scientific curricula. Since mathematics is a subject that cannot be bypassed nor easily replaced by experience in the field, mathematics staffs will be under special stress to carry this additional burden. Indeed it is not unlikely that the load of teaching elementary college mathematics will exceed our present one, including the present Army and Navy programs.

5. Advanced mathematics in engineering and scientific courses. I foresee also a considerable expansion at the upper end of the engineering and scientific curricula. Many young men will have completed the somewhat abbreviated or at least hurried courses called advanced engineering in the specialized service programs. These young men as well as many recent engineering graduates on leaving the army will see in the federal educational program an opportunity to continue their studies from an apprentice to a professional level. It would probably be idle here to try to fix the point at which preprofessional work in engineering ends and professional training begins. It is certainly true that in such professions as law, medicine, and the ministry, formal schooling does not end

after four years beyond the high school. Perhaps we are willing to entrust our rights, our health, and our souls, to the schools but not our material possessions. No more than we should expect every physician to be using basic science in research should we expect every professional engineer to employ mathematics at the research level. Yet, because of the essentially quantitative character of engineering work, we might not be too far off the mark in accepting the skill with which an engineer uses applied mathematics as one measure of his professional attainment. It has been repeatedly pointed out of late, notably by Thornton C. Fry in his report on mathematics in industry,* and by R. G. D. Richardson in this MONTHLY,† that the study of applied mathematics in America has not progressed as it has in certain foreign countries nor as has the study of pure mathematics here. It is perhaps a natural consequence of this that in many of our four- and five-year engineering schools the curricula are less mathematical and to an extent less professional in character than they might well be. Not in all but in many of them it is the tradition to consider mathematics as a tool subject taught on a relatively formal and mechanical basis. In such cases the special power of applied mathematics is largely lost. Perhaps the ideal in engineering training would be to combine the power of generalization, which implies the discernment of the essential in a problem, and which is so characteristic of modern American mathematics, with an interest in and familiarity with the materials of some field of technological application. I am inclined to believe that such considerations will ultimately lead away from the formal toward the more theoretical, that is, the ultimately more practical type of mathematics course in engineering colleges. Immediately after the war this tendency may not exist and may even be reversed for the courses in the first two years of the curriculum. This reversal will probably result from the lack of qualified teachers, the weak preparation of the mass of entering students, and from the desire of men already advanced in years and maturity of experience to pursue their training in as rapid and concentrated a form as possible.

But the war will itself have produced such advances in engineering practice that there will be a great demand for men of really superior training for the mere manufacture, installation and operation, let alone the design, of equipment.

Some days ago I talked with a young man, a recent first-year graduate student of mathematics, who is employed in one of the large government laboratories. The work he is doing is not concerned with the discovery of basic scientific facts, but is exceedingly practical in character and involves the application of familiar principles to strictly engineering problems. He was able to solve some of these problems by means of somewhat complicated differential equations, the application of potential theory, contour integration, and the use of Fourier transforms. He feels that, though he has had no engineering training,

* Research—A National Resource—II, VI, 4, pp. 268–288—A House Document, 77th Congress.

† Applied Mathematics and the Present Crisis. Vol. 50, No. 7, Aug.-Sept., 1943, pp. 415–423.

he is at an advantage as compared with others in the laboratory because he finds it natural to formulate a problem in mathematical language and to solve it mathematically. Another first-year graduate student of my acquaintance was employed by a company that manufactures intricate apparatus now used in the war. In this case approximations by means of orthogonal polynomials furnished the key to important improvements in equipment. These are merely two examples that many of you could doubtless duplicate many times. The point that I wish to emphasize is not the applicability of mathematics to engineering, which needs no emphasis. It is rather the fact that these were first-year graduate students and the techniques that they employed would be accessible to many young engineers with only a moderate increase in their mathematical background.

From these considerations I should expect a heavy increase in registration in our courses of intermediate and moderately advanced mathematics in engineering and scientific schools. I also expect a strong demand for additional or more extensive courses at this level, such as more courses in statistics with special emphasis on manufacturing and quality control, potential theory, advanced differential equations, theory of elasticity, group theory and fluid dynamics. How far and how promptly our depleted and overburdened staffs will be able to meet this demand is another story.

I should like to quote from a letter that I have just recently received from Colonel Andrews of the Reemployment Division of Selective Service, whom I have already mentioned. It seems to indicate that my own opinions are shared by those studying the problem from the point of view of employment. He writes: "Because of the unusual technical advances which are certain to occur following the war, all kinds of mathematicians, scientists, and research specialists will be in great demand. Such opportunities are already being discussed in the large industries and, eventually, the men in the services will be informed of the need for such specialists. It is believed that thousands of the young men will seek further training in these technical fields as soon as they are no longer needed in the armed forces. This program will call for a tremendous mathematics and science emphasis at the freshman and sophomore levels, especially, and also a continuation of these subjects in the junior and senior years. Perhaps the graduate work in these subjects will have the greatest challenge of their history. In order to meet these expanding demands, colleges and universities which have the proper staffs and laboratories should begin now the development of curricula to meet the needs."

6. Need of interdepartmental collaboration; liaison courses. What I have said applies equally well to the training of engineers and of other scientists. In both cases there are certain lessons to be learned from the military courses that we have taught. Some of these have required the closest collaboration between mathematicians and physicists. The pre-meteorological courses in mathematics, vectorial mechanics, and physics well illustrate the advantage of such collaboration in the organization of material. Especially in their earlier parts, a careful

ordering of topics supplied in one course the materials necessary in another in time for their utilization there, and avoided unnecessary duplications of material. I hope that with the return of peace these lessons of cooperation will not be forgotten. In our standard curricula there is usually sufficient time to allow a course in one subject to follow logically certain prerequisite courses in other fields. Thus a man-sized course in physics ordinarily follows at least a year of college mathematics. But probably we shall have many mature students seeking rapid advancement in rather narrow scientific or technical training. It may then be necessary and, in their cases, desirable to teach them their mathematics and physics concurrently instead of in sequence. Our experience with the war training courses has shown that in case of necessity this can be done with considerable success. Such procedures depend, however, on the closest and most sympathetic interdepartmental collaboration. I shall refer later to the other necessity of planning for courses that are less narrow and concentrated in design.

With the increased interest in applied mathematics after the war I expect a tendency toward what I may call "liaison" courses. By these I mean courses that do not lie wholly in one conventional field, but furnish contact between two or more departments. The pre-meteorological course in Vectors and Mechanics is an example. It formed a most useful link between mathematics and its applications. It was most interesting also in another respect. It showed the feasibility of introducing relatively advanced material early in the college course. In the "C" program it successfully presented to freshmen concurrently with the first course in calculus and after only twelve weeks of college mathematics, material of vector analysis that had usually been reserved for juniors and seniors. The early presentation of this material undoubtedly greatly strengthened the calculus course by giving greater reality to such notions as curvilinear velocity and acceleration. It may well be that other entire courses or special topics require less maturity than we have hitherto assumed and that they can be introduced advantageously earlier in the curriculum than we supposed, especially for students seeking rapid advancement in a special field.

7. Statistics. Another interesting type of liaison course is one in elementary statistics. An immense amount of statistical work will be required as a result of the war. To realize this one has to think only of such sample problems as the rehabilitation of veterans, job surveys, reconversion of industry, quality control in the manufacture of new products, distribution of food and other commodities to our own and foreign populations, taxation and debt refunding, social security, and the redistribution of populations here and abroad. And I believe that mathematics departments should give careful study to their responsibilities in training for this work.

We have probably all considered this problem; some but not all of us have reached successful solutions. I know of one university where beginning courses in statistics are taught in departments of sociology, psychology, educational psychology, economics, preventive medicine, agriculture, mathematics and probably some others. It is as if, in a college of engineering, the basic mathe-

matics were taught in separate departments such as physics, chemistry, electrical engineering, and mechanical engineering since each mistrusted the applicability of general mathematics to its own field. Yet, in statistics as elsewhere, it is the special power of mathematics to recognize the essence of a problem regardless of its immediate origin, to formulate quantitative definitions that really correspond to intuitive notions, to supply a tractable notation, and to analyze the relations existing between the elements. The inauguration of liaison courses in statistics involves not only the presentation of the theory common to all fields but also such interdepartmental cooperation that the results can really be utilized in the different departments. The problem is difficult, as we all know, but is likely to grow in importance. In relatively few institutions, I believe, has it been solved in a satisfactory way.

Not the least of the difficulties in connection with these liaison courses is that of securing instructors who have the ideal training already suggested for engineers, a thorough knowledge of fundamental mathematics combined with an interest in and real familiarity with some field of statistical or other application. In our graduate schools we should probably encourage or even require our students to become familiar with at least one field of applied mathematics as well as with their own line of specialization.

8. The need for general rather than limited courses. Before passing on to other matters I should like to make an observation suggested by these remarks on general or fundamental courses of statistics in contrast to courses limited to the applications in special fields. It has been remarkable and gratifying that in their training programs, the Army and Navy have sought mathematics. Not army mathematics nor navy mathematics, but fundamental mathematics. In their outlines of courses and in their lists of suggested materials of instruction, the services have very evidently appreciated that a general understanding of principles can more easily be applied to special situations than techniques learned only in connection with special problems can be transferred to unfamiliar areas. Of course, this does not mean that our teaching should deal only with general theorems and principles unrelated to any special problems that are familiar or exciting to the student. On the contrary, our courses should be as rich in varied and timely applications as we can possibly make them. Such applications provide the keenest motivation and the most satisfying means of clarification. Some of the books published during the emergency have made a genuine contribution just because they supplied a wealth of timely illustration. There have been relatively few, and most of them have been short-lived, of the meretricious publications designed to catch the reader with the bait of being, as they put it, "completely streamlined for the emergency," but written without regard to general principles or to the soundness that is the very strength of mathematics. Tendencies in this direction appeared early in the emergency, but received little encouragement from the armed services themselves. The most mathematically minded of navigators is not inclined to solve long problems in spherical trigonometry while navigating an airplane. But the Army and Navy

showed great appreciation of the value of a general mathematical background even for men in training for very specific tasks. They were glad to enlist the aid of the colleges in securing this background for their trainees, and were willing themselves to provide the special applications required in service. I believe that this is the case even in the Pre-Flight course. It is true that the content of this course is extremely elementary. That is a question of the level of instruction and the quality of the men selected, not of generality. In fact it is shockingly disturbing to find that at the college age it is necessary for the men to receive this type of instruction, which is so generally necessary that all should have received it as children.

How fortunate it is that the Army did not build its curricula too narrowly around a single limited objective, but preferred to supply fundamental training in the basic sciences is well illustrated in the recent developments in the pre-meteorological programs. What a tragic waste it would have been had all the mathematical and physical training been centered in a tight circle about immediate meteorological applications! As it was, once the rude shock of the suspension of the "A" course had been absorbed, the morale of the men came back again, for we could assure them in all good faith that their effort was not wasted and that what they had learned could be transferred with no loss whatever to many other interesting and useful tasks.

These thoughts also have their lessons for the period after the war. If the Army can go wrong in predicting the requirements for meteorological officers (and few of us would quarrel with it for overestimating rather than underestimating them), is it likely that errors will not be made in surveys of the requirements in various trades and professions? And can any counselling service be so perfect that it can foretell with certainty the best outlets for all of the individual men? To a great extent we must resist the pressure to make our courses too limited and specific. We must so imbue them with the content and spirit of real mathematics that their results can be transferred to any field of application or of thought.

9. The necessary and the unnecessary in mathematics. I should like to make another remark closely related to this one. Some of the persons engaged in planning for the education of returning veterans are insistent, and, in a measure, rightly so, that this education be practical and concentrated in form. They point out that a man in his twenties will be unable to afford a long time for continuing his studies and that he will demand a maximum of content in a minimum of time. I agree with such statements and I believe that for many of these returning soldiers we shall have to give highly concentrated courses, as I have said before. But I cannot follow these gentlemen quite so far as they would have us go. One of them, in referring to some of the wartime mathematics courses says, "Mathematics has had much of the unnecessary subject matter 'squeezed out,' and it is very likely that this condition in the future will maintain." If by this is meant that in the future we should scrutinize our subject

matter to see that it contributes to some really useful purpose, I agree. But I reserve the right to judge what is and what is not "unnecessary." And I should not consider subject matter unnecessary because it contributes to logical understanding and independence of thought instead of merely to formal technique. Nor should I consider a course too dilute merely because it covers the lapse of time necessary for some contemplation. By too much "squeezing" we may lose the life-giving juices and obtain indeed a desiccated product.

10. Placement problems. When our soldiers return, there will be great difficulty in placing them at suitable levels in our educational program. Our colleges should take steps at once to establish counselling systems adequate for the immense responsibility. It is to be hoped that these systems will not be entrusted to narrow groups of experts without a broad knowledge of the possibilities in the various fields, and that the basic sciences will be duly represented. Obviously, because of their maturity, the returning soldiers cannot be sent to high schools. But many of them will not have completed the prerequisites for the college courses they will wish to follow. We must share the responsibility of fitting these men into the educational scheme. In some wartime programs we have experienced the possibility of giving courses at elementary levels to prepare men for their collegiate work. Distasteful as it may be in some respects, I foresee the necessity of giving such courses for the returning veterans. I am not entirely sure that we should be wrong in making such courses available in peacetime to civilians. In foreign languages and in the natural sciences it is possible for college students to obtain beginning courses. In such courses it is possible for the students to proceed more rapidly than in high school and soon to press on the heels of those entering with one or two years of high school preparation. The chief danger in a similar procedure in mathematics would be in the fact that some high school advisers and administrators, knowing that these courses are available in college, would encourage pupils to omit mathematics from their high school programs. They would overlook the highly sequential nature of mathematics and not realize the necessity of beginning that subject as early in one's experience as possible. But the courses I have suggested may nevertheless be desirable solely as a means of rescue for the returning veterans and for the young civilians who, through poor advice or change of plans, may have omitted the high school mathematics necessary for their collegiate work.

11. Mathematics in general education. Up to this point I have spoken of mathematics chiefly in connection with its applications. Perhaps this is because these aspects have been stressed in the specialized training programs of the Army and Navy. There is a strong danger that these aspects are so obvious now that many educators will come to think of mathematics as useful only in its most crassly practical connections. There are some who have that opinion now. In some quarters there will be a strong reaction away from the technological toward general education. They ask, as people are asking in so many realms of life, of what avail will it be to win the war unless a better world emerges from it.

To help make this better world, we must educate our young people in the social sciences, in literature, in the fine arts, in philosophy. We must give them a general education. To all of this I agree, of course. The danger lies in the very fact that mathematics has become so conspicuous for its immediate practical necessity that its utility in general education will be forgotten.

In one instance, at least, the Army gave a sort of left-handed recognition to mathematics as a tool of general education. When asked why mathematics had to be included in a certain one of its training programs, the officer in charge replied, "Well, mathematics matures the men." Sometimes, I am sure, we have felt that it not merely matured the men but made them grow old before their time. What the officer meant, of course, was that mathematics is especially useful in the development of the general qualities that it values in its specialized personnel—qualities, perhaps, of clear, disciplined, hard-headed thought and clear expression.

The proposed federal program of subsidization of education that I have already outlined provides for general education as well as for vocational or professional training. If we are to do our share in carrying out the program, we must keep vividly in our own minds how mathematics can contribute to the objectives of a general education. And we must keep it in the minds of administrators in high schools and colleges and of the public as well. Of all the topics that I have mentioned this evening this one of the place of mathematics in general education lies as close as any to my heart. I had originally intended to make it the central theme of this address. It was my thought to recall the objectives of a college education and to examine the manner and the extent to which mathematics can definitely and in some respects uniquely contribute to attaining those objectives. The subject is too important and too large for the cursory treatment that I could give it now. With the hope that it will be discussed at length in some more suitable time and place, I shall assume that, as mathematicians, you share my feelings in this matter, and pass on to some of the immediate implications.

If it is true that we shall have many students returning from the service or coming to us from the high schools for a program of general education to which mathematics can make valuable contributions, then we must see to it that those contributions are made. Besides our extensive mathematical curricula, which secure many of the advantages of general education while training the student in his specialty, we must provide elementary terminal courses for this purpose. These courses, if less complete in the techniques necessary for continuation courses, should be rich in essential ideas. They should insist on precision and clarity of language. They should reveal mathematics as a logical structure arising from simple and natural assumptions, explicitly recognized, and leading to important and interesting results. They should indicate the necessity in any system of thought of exact definition of terms and provide examples of definitions that clarify and correspond to one's intuitive notions. We must not lead the student to believe that mathematics is vague or inconclusive by attempting

to include too great a range of material or any material too remote from his experience. The time was when it was believed that the world could be saved by education—any education. But we have learned that education built on authority alone can work as efficiently for evil as for good. If education is to contribute to our national welfare, it must conduce to independence of thought, by which the individual is able to examine for himself the assumptions and conclusions behind dogma. I am convinced that the courses I have in mind can aid materially in establishing desirable habits of scientific thinking. As you know, such courses already exist in many institutions. I believe that we should make still further provision for them.

12. Shortages of staff. As I have surveyed these tasks that lie ahead of us, I have been haunted by a certain fear. It is the fear that we shall not have the personnel properly trained to carry the great load. As I have indicated, there is every prospect that at the freshman and sophomore levels our burden of teaching, far from decreasing, will actually increase after the war. Added to this, we should have more students than ever before at the higher levels, where the war-time load is most conspicuous by its absence. Just who is to do this teaching? Some of our staff, away on war duty, will return. But many who have tasted the flesh-pots of industry will not wish to go back to the genteel poverty of college teaching. Many teachers lent to us from other fields will be called back to duty in their own departments. Many scientists have been sacrificing their own interests to the common good by teaching instead of devoting their energy to research. It would dry up the very sources of progress to ask them to continue this sacrifice indefinitely.

And what of the supply of replacements? Almost we can say there isn't any. The normal crop of young students has been mostly absorbed into the Army or Navy or other war service. Selective Service, it is true, has recognized mathematics as a field in which there exists a critical shortage. But so far as students are concerned this has resulted chiefly in the deferment of undergraduates. Most graduate students, under the pressure of war-training courses and the attractions of high salaries, have been teaching instead of studying. A full-time graduate student or a part-time assistant is a *rara avis*.

13. The training of mathematicians. Of all our future tasks, and I have referred to a number of them this evening, perhaps the most pressing, most important, and most difficult is this one of the training of mathematicians for our college and university staffs. The plant must be erected before munitions can be produced. We can begin even now to do a little work on the problem. Just the other day a former high school teacher whose previous maximum salary for a nine-month year was $1620 was offered a college position at $3600 for twelve months. Such opportunities are tempting. But I believe that our younger staff should be informed of the opportunities that will exist after the war, and should be encouraged to take every present chance to carry on their studies and their research. The present load of teaching must be carried on. However,

young people of special promise should be encouraged to do their teaching under conditions that offer some opportunity for study. So far as the slightest break in the wartime schedules permits, we should provide graduate courses and encourage our young teachers and students to take them. If we realize the prime necessity of this part of our program, we can do something to relieve the situation, but only in a piecemeal and inadequate way.

Under the present war training contracts, institutions are able to offer attractive salaries. It is not likely that the scholarships offered under any program of federal subsidization after the war will be large enough wholly to pay the costs of instruction. Colleges and universities will be hard put to it to offer salaries commensurate with the demand. It is not too early to begin to bring the prospective needs of education before the public with the hope of winning financial support. Even if such a campaign of public education were successful, it would not solve our problem. Money can attract mathematicians, but it cannot make them.

14. A possible source of teachers. There is only one large source of partly trained manpower. That is in the armed services themselves and especially in their present educational programs. The Army and Navy should recognize their responsibility to the servicemen after demobilization. If it is true, as I believe it is, that the scarcity of teachers will be as critical for these men as the scarcity of physicians, it would seem reasonable that the Army should immediately undertake a long-range program of training scientists similar to its program of medical training. If, for example, the best fifteen or twenty per cent of graduating pre-meteorology students were to be offered a year or more of concentrated but not too narrow training in mathematics and its applications, wonders could be done in creating a supply of men capable of giving at least elementary college instruction. This particular plan is fantastically improbable. But it might not at all be a waste of effort for the officers of the mathematical organizations to present to the A.S.T.D. the desperate necessity of some such scheme of long-range advanced instruction in mathematics.

15. The unity of the mathematical program at all levels. I should like to mention just one other thing before I close. I refer to the essential unity of the mathematical program at all levels from the junior high school, through the stages of high school, junior college, and college, to the graduate school. Research and graduate teaching vitalize college work. The colleges stimulate the teaching and set the standards for the high schools; in turn they are dependent on the high schools for both the number and quality of their students. Because of the extremely sequential nature of our subject this unity is even more important in mathematics than it is in most fields. Students who have been badly advised or badly taught in high school have special difficulty in repairing the damage to their mathematical training. I do not need to recall here the trends in attitude toward high school mathematics. It is enough to say that many educators expect and some of them desire that recent trends will be reversed and those of a

few years ago will be renewed. We must furnish what help we can to the high school program. Quite apart from what we are tempted to feel are malicious attitudes toward mathematics, we must educate ourselves to the very genuine difficulties in the high school mathematical curriculum. Some of these difficulties result from the individual differences in ability of students; others from their differences in purpose. Others arise from an ignorance of the number of paths barred to them by an early neglect of mathematics.

The Mathematical Association of America has always had close contacts with the other mathematical organizations. We have held joint meetings with both the American Mathematical Society and with the National Council of Teachers of Mathematics. I can only urge that in the highly critical period just after the war, as an organization and as individuals we participate even more actively in the problems of our high school colleagues. Only by considering the entire mathematical program as a unit can we achieve the strength that comes from unity.

In this survey of what might be called "War's Math and Aftermath," I have attempted to consider the general magnitude of the task ahead, with some of its opportunities and problems. At moments I have stated my own opinions while quite aware that many of you will not agree with them. If I have spoken dogmatically at certain points, I did so only for purposes of economy in presentation and not because I felt that I had direct inspiration in the matter. I do believe that it is time for the subject to be brought before the house, and I hope that my own speculations may give rise to profitable discussion.

CARDIOIDS ASSOCIATED WITH A CYCLIC QUADRANGLE

V. O. McBRIEN, Hamilton College

1. Introduction. If a quadrangle is inscribed to a circle, then for any given line l in the plane of the quadrangle there are four orthopoles, one with respect to each of the four triangles formed by the vertices. S. Kantor has proved that the four orthopoles lie on a line which we shall call the Kantor line of the line l with respect to the cyclic quadrangle [1]. It was later proved that the quadrangle need not be cyclic but can be *any quadrangle* in the plane [2], the line in the general case being called "the orthopolar line of the line l."

Using circular coordinates, the equation of the Kantor line is

$$2az + 2s_4\bar{a}\bar{z} - a^2 - as_1 - \bar{a}^2 s_4 - \bar{a}s_3 = 0 \tag{1}$$

where s_i $(i=1, 2, 3, 4)$ are the elementary symmetric functions of the unit vectors t_1, t_2, t_3, t_4 to the vertices of the cyclic quadrangle, and $z/a+\bar{z}/\bar{a}=1$ is the conjugate equation of the given line l. The envelope of Kantor lines of a pencil of lines on the fixed point z_0 with respect to the cyclic quadrangle is [2]

$$z = \frac{z_0}{2} + \frac{s_1}{2} - \bar{z}_0 t - \frac{z_0 s_4}{2t^2}. \tag{2}$$

This is the map equation of a deltoid* whose size is one-half the distance from the center of the pencil to the circumcenter.

O. J. Ramler has derived the equation of the ellipse locus of orthopoles of the pencil of lines on a point z_0 with respect to a triangle [3]. It is

$$z = \frac{z_0}{2} + \frac{\alpha_1}{2} - \frac{1}{2}\left(\bar{z}_0 t + \frac{\alpha_3}{t}\right) \tag{3}$$

where α_i $(i=1, 2, 3)$ are the symmetric functions of t_1, t_2, and t_3. In all, four such ellipses can be obtained by taking three at a time of t_1, t_2, t_3, t_4. The clinant of this ellipse is

$$\frac{dz}{d\bar{z}} = \frac{\alpha_3(\alpha_3 - \bar{z}_0 t^2)}{\alpha_3 z_0 - t^2} \tag{4}$$

while the clinant of the deltoid is

$$\frac{dz}{d\bar{z}} = \frac{s_4}{t}. \tag{5}$$

By use of the cubic in t

$$t^3 - \frac{t_i}{\bar{z}_0} t^2 - \frac{s_4}{t_i \bar{z}_0} t + \frac{z_0 s_4}{\bar{z}_0} = 0 \qquad (i = 1, 2, 3, 4), \tag{6}$$

intersections of each of the ellipses (3) with the deltoid are obtained. Now if we set the clinants of the deltoid and ellipse equal we obtain the same cubic (6). If the coefficients of a cubic satisfy certain conditions, then all the roots of the cubic correspond to points of the unit circle, or at least one does. It can be shown that the coefficients of the cubic (6) do satisfy these conditions [4]. Thus, each ellipse is tangent to the deltoid at least once.

It is the purpose of this paper to investigate the number of the points of tangency of these curves and to show how the number depends upon the position of z_0 with reference to four cardioids associated with the quadrangle. We shall consider also some of the properties of these cardioids and their relation to the quadrangle.

2. The cubic and its discriminant. If in the cubic (6), with $i=1$, $z_0=T$, a point on the unit circle, then (6) becomes

$$(t - t_1 T)\left(t^2 - T\frac{s_4}{t_1}\right) = 0. \tag{7}$$

The roots of this equation are Tt_1, $\pm\sqrt{TS_4/t_1}$. If T equals S_4/t_1^3, two of the roots

* A deltoid is a three-cusped hypocycloid; its "size" is the radius of its inscribed circle.

coincide and the ellipse is tangent at a cusp of the deltoid.

In general, to find the region of z_0 where the ellipse will have one, two, or three points of tangency with the deltoid we first obtain the discriminant, Δ, of the cubic by the ordinary method.* Upon expanding and simplifying,

$$\Delta = \frac{s_4}{\bar{z}_0^4}\left(\frac{4s_4^2\bar{z}_0}{t_1^3} + s_4 - 27z_0^2\bar{z}_0^2 s_4 + 18z_0\bar{z}_0 s_4 + 4z_0 t_1^3\right). \tag{8}$$

3. The cardioid. Turning from the discriminant, let us find the conjugate equation of a cardioid with a cusp at $z=1$, when $t=1$. The map equation is [5, 6]

$$z = 2t - t^2. \tag{9}$$

Eliminating t between (9) and its conjugate we find the conjugate equation of this cardioid is

$$z^2\bar{z}^2 - 6z\bar{z} + 4z + 4\bar{z} - 3 = 0. \tag{10}$$

The second factor of the discriminant (8) resembles equation (10) except for a transformation constant which is readily found. If the discriminant is set equal to zero, however, we must account for the factor $s_4/\bar{z}_0^4$. Since $s_4 = t_1t_2t_3t_4$, $|s_4| = 1$. If $\bar{z}_0$ were zero, then the ellipse becomes

$$z = \frac{\alpha_1}{2} - \frac{\alpha_3}{2t} \tag{11}$$

where $\alpha_1 = t_2+t_3+t_4$ and $\alpha_3 = t_2t_3t_4$. The deltoid would be reduced to the point $s_1/2$. The locus (11), which is the nine-point circle of triangle $t_2t_3t_4$ passes through $z=s_1/2$ when $t=-\alpha_3/t_1$.

Thus we may say even here that the ellipse is "tangent" to the "deltoid" $z=s_1/2$. It is a well known theorem that the nine-point circles of the four triangles formed by four concyclic points are concurrent. The map equation $z=1/2(t_1+t_2+t_3+t)$ shows that the circle passes through $s_1/2$ when $t=t_4$.

Now setting the discriminant Δ equal to zero, omitting the factor $s_4/\bar{z}_0^4$ and replacing z_0 by z, we have

$$z^2\bar{z}^2 - \frac{2}{3}z\bar{z} - \frac{4}{27}\frac{t_1^3}{s_4}z - \frac{4}{27}\frac{s_4}{t_1^3}\bar{z} - \frac{1}{27} = 0. \tag{12}$$

By inspection of (10) and (12) it is clear that the transformation $x = -3t_1^3z/s_4$ will send the latter into the former. Thus, the number of the points of tangency of ellipse and deltoid depends upon the position of z_0 relative to the cardioid given by equation (12).

4. The cusps and the cusp tangents. Solving the equation of the cardioid (12) with the equation of the circumcircle of the quadrangle, $z\bar{z}=1$, results in the equation

* See Dickson's Elementary Theory of Equations.

$$z^2 - 2\frac{s_4}{t_1}z + \frac{s_4^2}{t_1^2} = 0 \tag{13}$$

which has a double root, $z = s_4/t_1^3$. Hence the cardioid is tangent to the circumcircle. If we call this vector τ, the equation of the diameter $O\tau$, O being the circumcenter, is

$$z - \frac{s_4^2}{t_1^6}\bar{z} = 0. \tag{14}$$

Substituting $z = s_4^2\bar{z}/t_1^6$ in equation (12) we may obtain the intersections of this diameter with the cardioid. For this we have the equation

$$z^4 - \frac{2}{3}\frac{s_4^2}{t_1^6}z^2 - \frac{8}{27}\frac{s_4^3}{t_1^9}z - \frac{1}{27}\frac{s_4^4}{t_1^{12}} = 0. \tag{15}$$

By synthetic division, $z = s_4/t_1^6$ is shown to be a root, and, furthermore, the vector $z = -s_4/3t_1^3$ is a triple root of (15). Hence it is the vector to the cusp of the cardioid. This point is at a distance from O equal to one-third the diameter of the circumcircle.

Now for every point z_0 in the plane of the quadrangle there are four cardioids, each associated with the deltoid and one of the four ellipses mentioned in Section 2. The cusps therefore lie on a circle concentric with the circumcircle, the latter having a radius three times as large as the former. Furthermore, each cardioid is tangent to the circumcircle at the points

$$\frac{s_4}{t_1^3}, \quad \frac{s_4}{t_2^3}, \quad \frac{s_4}{t_3^3}, \quad \frac{s_4}{t_4^3}.$$

Since all four cusps lie on the circle $|z| = 1/3$, then when $|z_0| < 1/3$ the point z_0 lies inside each of the four cardioids. To obtain three distinct points of tangency we let $|z_0| \geqq 1$ since then the point lies outside the cardioid which is tangent to the circumcircle.

References

1. Kantor, S., Ueber eine Gattung merkwuerdiger Geraden und Punkte bei vollstaendigen n-Ecken auf dem Kreise, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Classe, Staatsdruckerei, Wien, 78, 2–3, 1828, p. 193.

2. McBrien, V. O., On some systems of orthopolar and Kantor lines of lines referred to a quadrangle, The Catholic University of America Press, Washington, D. C., 1942.

3. Ramler, O. J., The orthopole loci of some one-parameter systems of lines referred to a fixed triangle, this MONTHLY, 27, 1930, p. 130.

4. Ramler, O. J., Quadratic and Cubic Equations, this MONTHLY, 50, 1943, pp. 507–509.

5. Morley, Frank and Morley, F. S., Inversive Geometry, Ginn and Co., London, 1933, p. 239.

6. Wagner, P. S., An extension to Clifford's Chain, Amer. Journal of Mathematics, 51, 1929, p. 473.

AFFINE GEOMETRY OF CONVEX QUARTICS

DOUGLAS DERRY, University of Saskatchewan

1. Introduction. Let C be a curve defined by the equation $f(x, y) = 1$, where $f(x, y)$ is a homogeneous polynomial of degree 4. Let P be a parallelogram with one vertex at the origin and the other three vertices on C. C and P will retain these meanings throughout this article. If C is convex, the area of P is a single valued continuous function of the inclination of one of its sides with the x-axis. The following note uses elementary propositions in affine geometry* to locate those parallelograms P of maximum or minimum area for convex quartics C. Mordell† has recently solved the problem by other methods in the case where the equation of C is $Ax^4+By^4=1$.

2. Preliminary area theorems. We shall now derive some theorems which will be used in the proof of our main result, which appears in Theorem 5.

THEOREM 1. *Let the points* $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$ *be on the curve* C. *Let* $m_1(=(1/r(dr/d\theta))$ *be the cotangent of the angle from the radius vector joining the points* $(0, 0)$ *and* $(1, 0)$, *to the tangent to the curve at the latter point. Let* m_2, m_3 *be analogously defined for the points* $(\pm 1/2, \sqrt{3}/2)$. *If the points,* $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$ *be vertices giving a maximum or minimum area for* P, *then* $(m_1+m_2+m_3)/3+m_1m_2m_3=0$.

Proof. Let (r_1, θ_1), (r_2, θ_2), (r_3, θ_3), $\theta_1<\theta_2<\theta_3$, be the polar coordinates of the three points of P which lie on C. Clearly,

$$\text{Area of } P = r_2r_1 \sin(\theta_2-\theta_1) = r_3r_1 \sin(\theta_3-\theta_1) = r_3r_2 \sin(\theta_3-\theta_2).$$

If P has a maximum or a minimum area, we have, by logarithmic differentiation,

$$\frac{1}{r_2}\frac{dr_2}{d\theta_2}d\theta_2 + \frac{1}{r_1}\frac{dr_1}{d\theta_1}d\theta_1 + \cot(\theta_2-\theta_1)(d\theta_2-d\theta_1) = 0,$$

$$\frac{1}{r_3}\frac{dr_3}{d\theta_3}d\theta_3 + \frac{1}{r_1}\frac{dr_1}{d\theta_1}d\theta_1 + \cot(\theta_3-\theta_1)(d\theta_3-d\theta_1) = 0,$$

$$\frac{1}{r_3}\frac{dr_3}{d\theta_3}d\theta_3 + \frac{1}{r_2}\frac{dr_2}{d\theta_2}d\theta_2 + \cot(\theta_3-\theta_2)(d\theta_3-d\theta_2) = 0.$$

Let us specialize P so that it becomes the parallelogram $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$. By using the fact that this parallelogram is composed of two equilateral triangles and by replacing the derivatives by their equivalent cotangent expressions, we have

---

* W. C. Graustein, Introduction to Higher Geometry, New York, 1930, pp. 179–186.

† L. J. Mordell, Lattice points in the region $|Ax^4+By^4| \leq 1$, Journal London Math. Soc., 16, 1941, pp. 152–156. The reader is also referred to this article for an immediate application of our problem to lattice theory.

$$\left(m_1-\frac{\sqrt{3}}{3}\right)d\theta_1+\left(m_2+\frac{\sqrt{3}}{3}\right)d\theta_2=0,$$
$$\left(m_1+\frac{\sqrt{3}}{3}\right)d\theta_1+\left(m_3-\frac{\sqrt{3}}{3}\right)d\theta_3=0,$$
$$\left(m_2-\frac{\sqrt{3}}{3}\right)d\theta_2+\left(m_3+\frac{\sqrt{3}}{3}\right)d\theta_3=0.$$

We eliminate the differentials and obtain

$$\begin{vmatrix} m_1-\frac{\sqrt{3}}{3} & m_2+\frac{\sqrt{3}}{3} & 0 \\ m_1+\frac{\sqrt{3}}{3} & 0 & m_3-\frac{\sqrt{3}}{3} \\ 0 & m_2-\frac{\sqrt{3}}{3} & m_3+\frac{\sqrt{3}}{3} \end{vmatrix}=0.$$

After expanding this determinant and dividing by 2, we have $m_1m_2m_3+(m_1+m_2+m_3)/3=0$. Thus the theorem is proved.

We shall now use the concept of conjugate diameters. The definition of a pair of these diameters, as given in the theory of conics, may be extended immediately to any convex oval.

THEOREM 2. *If P has a maximum or a minimum area, then either one of the sides of P or one of its diagonals lies along one of a pair of conjugate diameters of C.*

Proof. If we subject a curve to an affine transformation, conjugate diameters are mapped into conjugate diameters and areas remain unchanged but for a constant multiple. It will, therefore, be sufficient to prove the theorem for C after it has been subjected to an arbitrary affine transformation. Accordingly, let C' be the affine map of C in which P is mapped into the parallelogram $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$.

C' being a fourth order curve with equation $f(x, y)=1$, its polar equation may be written in the form

$$\frac{1}{r^4}=A_0+A_2\cos 2\theta+A_4\cos 4\theta+B_2\sin 2\theta+B_4\sin 4\theta.$$

If we substitute the coordinates of the points $(1, \pi/3)$, $(1, 2\pi/3)$ into the above equation, the two relationships show that $B_2=B_4$.

By differentiating the equation of C', we have

$$-\frac{4}{r^5}\cdot\frac{dr}{d\theta}=-2A_2\sin 2\theta-4A_4\sin 4\theta+2B_2\cos 2\theta+4B_2\cos 4\theta.$$

By substituting the three sets of coordinates, $(1, 0)$, $(1, \pi/3)$, $(1, 2\pi/3)$, into this relation we get for the three cotangents of Theorem 1:

$$-4m_1 = 6B_2,$$
$$-4m_2 = -\sqrt{3}A_2 + 2\sqrt{3}A_4 - 3B_2,$$
$$-4m_3 = \sqrt{3}A_2 - 2\sqrt{3}A_4 - 3B_2.$$

We see that $m_1+m_2+m_3=0$. Hence, by Theorem 1, at least one of the cotangents m_1, m_2, m_3 vanishes. Without restriction in generality we may assume this to be m_1. Otherwise, by a rotation about the origin, we may bring the point with the vanishing cotangent into the polar axis. Now consider the new parallelogram, $(0, 0)$, $(1, 0)$, $(\pm 1/2, \sqrt{3}/2)$. This would have the same cotangents, in different order, and would likewise have a maximum or a minimum area. Clearly, in every event, a side or diagonal of the original parallelogram now lies along the polar axis. Because of the vanishing cotangent, we have $B_2=0$ and, consequently, $B_4=0$. Hence the equation of C' has the form

$$\frac{1}{r^4} = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta.$$

This equation shows the curve to be symmetric with respect to the lines $\theta=0$ and $\theta=\pi/2$. Hence these two lines are a pair of conjugate diameters and the theorem is established.

3. Some theorems in affine geometry. The following theorems which we use in the proof of Theorem 5, may be of some interest on their own account.

Let $f(x, y)=1$ be the equation of a fourth order convex curve C. Assuming the first member to be homogeneous, we have a real factorization

$$f(x, y) \equiv (ax^2 + hxy + by^2)(cx^2 + kxy + dy^2).$$

We define this factorization to be *normalized* if the areas of the two ellipses, $ax^2+hxy+by^2=1$, $cx^2+kxy+dy^2=1$, are equal. A normalized factorization is unique.

Except in the trivial case in which they coincide, the two component ellipses of a normalized factorization intersect in the four vertices of a parallelogram. We define this parallelogram to be the *critical parallelogram* of the curve C. An affine transformation maps the critical parallelogram into the critical parallelogram of the image of C.

THEOREM 3. *Except in the case in which it degenerates into an ellipse, a convex fourth order curve C has exactly two pairs of conjugate diameters: a pair along the diagonals of its critical parallelogram and a pair parallel to the sides of this parallelogram.*

Proof. Let $f(x, y)=1$ be the equation of a fourth order convex curve C. The two factors of the normalized factorization of $f(x, y)$ are distinct if we exclude

the case in which the curve degenerates into an ellipse. We may map the two component ellipses of this normalized factorization by a suitable affine transformation so that they become congruent to each other and with their major and minor axes along the coordinate axes. The normalized factorization of the resulting equation takes the form

$$(ax^2 + by^2)(bx^2 + ay^2) = 1. \tag{1}$$

The critical parallelogram is now a square with sides parallel to the coordinate axes and with the lines $y = \pm x$ as diagonals. Clearly the curve is symmetric with respect to the lines $y=0$, $x=0$, $y=\pm x$. Consequently the line pairs, $(y=0, x=0)$, $(y=\pm x)$, are conjugate diameters.

We now prove that these are the only conjugate diameters. Let us map any arbitrary pair of conjugate diameters into the coordinate axes by an affine transformation and consider the normalized factorization,

$$(ax^2 + hxy + by^2)(cx^2 + kxy + dy^2) = 1,$$

of the resulting curve. The condition for the axes to be conjugate diameters is that the coefficients of the terms involving x^3y and xy^3 vanish; *i.e.*,

$$ak + ch = 0,$$
$$bk + dh = 0.$$

We distinguish two cases: 1, $ad-bc=0$; 2, $h=k=0$.

As none of a, b, c, d may vanish without introducing a real linear factor into $f(x, y)$, in which case the curve would not be convex, we have, in Case 1,

$$\frac{a}{c} = \frac{b}{d} = r \text{ (say)}.$$

We deduce $a=cr$, $b=dr$ and $h=-kr$. The factorization now becomes

$$(crx^2 - krxy + dry^2)(cx^2 + kxy + dy^2) = 1.$$

By hypothesis this factorization is normalized, the component ellipses thus having equal area; therefore $r=1$. It follows easily that these ellipses intersect along the coordinate axes and so the conjugate diameters lie along the diagonals of the critical parallelogram of the curve.

In Case 2 the axes of the ellipses lie along the coordinate axes and it follows that they, and therefore the conjugate diameters, are parallel to the sides of the critical parallelogram of the curve. Thus we have located all the conjugate diameters and the theorem is proved.

THEOREM 4. *By a suitable affine transformation any finite fourth order curve, $f(x, y)=1$, with homogeneous first member, may be mapped into a curve whose equation is*

$$x^4 + kx^2y^2 + y^4 = 1, \qquad 2 \leqq k.$$

Those curves with $k \leqq 6$ are convex, those with $k > 6$ concave.

Proof. In equation (1) let $k=a^2+b^2$. If we impose the additional condition that each of the ellipses (of equation (1)) have area π, it follows that $ab=1$. Equation (1) may now be written

$$x^4 + kx^2y^2 + y^4 = 1.$$

As $a^2+b^2 \geqq 2ab=2$, we have $k \geqq 2$. Thus the first part of the theorem is established.

The above equation in polar coordinates becomes

$$\frac{1}{r^4} = \cos^4\theta + \sin^4\theta + k\sin^2\theta\cos^2\theta.$$

This is equivalent to

$$\frac{1}{r^4} = 1 + \frac{k-2}{4}\sin^2 2\theta.$$

Let $v=(k-2)/4$. To prove the remainder of the theorem it is sufficient to show the curves are convex if, and only if, $0 \leqq v \leqq 1$.

Let $h>0$. The area of the sector bounded by the curve and the radius vectors at θ and $\theta+h$ is

$$\frac{1}{2}\int_0^h \frac{dt}{\sqrt{1+v\sin^2 2(\theta+t)}}.$$

The curve will be convex if this area is not less than that of the triangle formed by joining the ends of the radius vectors for arbitrarily small h; *i.e.*, if

$$\int_0^h \frac{dt}{\sqrt{1+v\sin^2 2(\theta+t)}} \geqq \frac{\sin h}{\sqrt[4]{(1+v\sin^2 2\theta)(1+v\sin^2 2(\theta+h))}}. \tag{2}$$

A straightforward calculation gives the Taylor expansion

$$\int_0^h \frac{dt}{\sqrt{1+v\sin^2 2(\theta+t)}} - \frac{\sin h}{\sqrt[4]{(1+v\sin^2 2\theta)(1+v\sin^2 2(\theta+h))}} = \frac{h^3}{6}\frac{2v+1-(2v+v^2)\sin^2 2\theta}{(1+v\sin^2 2\theta)^{5/2}} \cdots.$$

As this is clearly positive for $v<1$, the inequality (2) holds. Hence the convexity of the curves is established for $0 \leqq v<1$; *i.e.*, for $2 \leqq k<6$. The inequality also proves the convexity of the curve for which $k=6$, except at the points for which $\sin^2 2\theta=1$. As we may approach these points by convex curves with $k<6$, lying outside of the curve for which $k=6$, this curve cannot be concave at these points and so must be convex throughout. For $v>1$ the coefficient of h^3 is negative for $\sin^2 2\theta=1$. Thus these curves are concave. The theorem is now completely proved.

4. The main result. We are now in a position to establish the result indicated in the introduction.

THEOREM 5. *For a convex curve C of fourth order, the maximum value of the area of the parallelogram P is attained when one of its sides or diagonals coincides with a diagonal of the critical parallelogram of the curve, and the minimum value is attained when one of the sides or diagonals of P is parallel to a side of the critical parallelogram.*

Proof. It follows from Theorem 4 that it is sufficient to prove this for the curves,

$$x^4 + kx^2y^2 + y^4 = 1, \qquad 2 \leqq k \leqq 6.$$

The critical parallelogram of each of these curves is a square with sides parallel to the coordinate axes. Because of the symmetry of the curves about the lines $y=0$, $x=0$, $y=\pm x$, maxima and minima occur when a side or diagonal of P is along one of these lines. Theorems 2 and 3 show that these exhaust every possibility. It remains to distinguish maxima and minima. Let us calculate the area A_1 of the parallelogram P with side along the x-axis. An elementary calculation shows the square of the altitude to be $(\sqrt{k^2+60}-k)/8$. As the base of the parallelogram is 1, we obtain $A_1^2=(\sqrt{k^2+60}-k)/8$. If A_2 denote the area of the parallelogram P with side along the line $y=x$, a similar calculation, after rotating the coordinate axes through 45°, gives

$$A_2^2 = \frac{k - 6 + 4\sqrt{k^2 + 3k + 6}}{(2 + k)^2}.$$

For $k=2$ the curve is a circle and the two values A_1, A_2 must be equal, which fact is easily verified. However they can never be equal for any other value of k, for then the maximum and minimum values would be the same; consequently the area of P would be a constant and so a maximum everywhere. It would then follow from Theorem 2 that every diameter would be a conjugate diameter in contradiction to Theorem 3.

However $A_1^2-A_2^2$ is a continuous function of k and if it changed signs within the interval $2<k\leqq 6$, a value of k would exist in this interval for which $A_1=A_2$, in contradiction to what we have proved above. Consequently one of these is a maximum, the other a minimum for all values of k. By actual calculation we find, for $k=4$,

$$A_1^2 = .58\cdots, \qquad A_2^2 = .59\cdots.$$

Hence A_2 is a maximum, A_1 a minimum for all values of k, and the problem is completely solved.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE EIGHT-POINT CIRCLE AND THE NINE-POINT CIRCLE

LOUIS BRAND, University of Cincinnati

It has apparently never been noted that the famous theorem on the *nine-point circle* is merely a special case of a very simple theorem on the *eight-point circle;* namely

When the diagonals of a plane quadrilateral are perpendicular, the mid-points of its sides, and the feet of the perpendiculars dropped from the mid-points on the opposite sides, all lie on a circle described about the mean center of the vertices.

Proof. In the quadrilateral $ABCD$ let P, Q, R, S denote the mid-points of the sides AB, BC, CD, DA (Fig. 1). Then $PQRS$ is a parallelogram whose sides are parallel to the diagonals AC, BD of the quadrilateral. When AC and BD

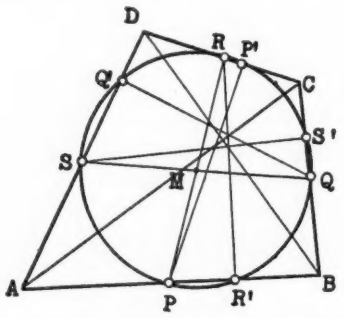

FIG. 1

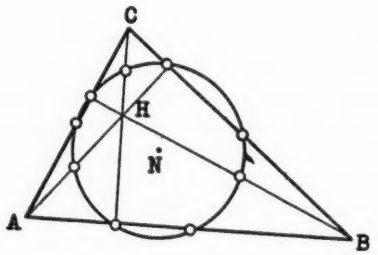

FIG. 2

are perpendicular, $PQRS$ is a rectangle; its diagonals PR, QS bisect each other at M, the mean-center of the points A, B, C, D (their centroid when each point has the weight 1). Therefore P, Q, R, S lie on a circle centered at M; and if perpendiculars from P, Q, R, S are dropped upon the sides opposite, their feet P', Q', R', S' will also lie on this circle. Thus the theorem on the eight-point circle is proved.

In the figure formed by a triangle ABC and its three altitudes meeting at the orthocenter H (Fig. 2), the three quadrilaterals $ABCHA$, $BCAHB$, $CABHC$ all have perpendicular diagonals. Their three eight-point circles (six-point circles in this case) coincide. The eighteen (3×6) points on this circle reduce to nine

as each point appears twice. The common eight-point circle of the quadrilaterals, about the mean center of A, B, C, H, is the nine-point circle of the triangle.

THEOREM. *For any triangle ABC whose orthocenter is H, a circle whose center is the mean center of A, B, C, H, passes through the nine points: the mid-points of the sides, the feet of its altitudes, and the mid-points of the segments joining H to the vertices.*

The center N of the nine-point circle is the centroid of A, B, C, H when each point has the weight 1. Now if G is the centroid of A, B, C, namely the point of intersection of the medians of the triangle ABC, we may also say that N is the centroid of G with the weight 3 and H with the weight 1. Therefore N divides the segment HG in the ratio 3/1.

NOTES ON THE GEOMETRY OF THE TRIANGLE

O. S. ADAMS, U. S. Coast and Geodetic Survey

If we use conjugate coordinates to denote the vertices of the triangle by turns t_1, t_2, t_3 of the circumcircle, the equations of the sides become

$$x + t_1t_2\bar{x} = t_1 + t_2,$$
$$x + t_1t_3\bar{x} = t_1 + t_3,$$
$$x + t_2t_3\bar{x} = t_2 + t_3.$$

The t's are complex numbers with absolute value of unity. Denote their elementary symmetric functions by S_1, S_2, S_3. If we add the three equations of the sides of the triangle we get

$$3x + S_2\bar{x} = 2S_1$$

in which x is the coordinate of a definite point in the triangle. By taking the conjugate of this equation, remembering that $\overline{S}_2 = S_1/S_3$ and $\overline{S}_1 = S_2/S_3$, we find

$$3S_3\bar{x} + S_1x = 2S_2.$$

By eliminating $\bar{x}$ from these two conjugate equations, we get

$$x = \frac{6S_1S_3 - 2S_2^2}{9S_3 - S_1S_2}.$$

This is the symmedian point of the triangle, which is usually denoted by K. At once the question arises, "why does this procedure give the symmedian point of the triangle?"

If we have an equation of a line in the self conjugate form, we can substitute in it the coordinate of a point not on the line. What does this procedure yield? If we take a point p in the plane of the triangle but not on a given side, such as $x + t_1t_2\bar{x} = t_1 + t_2$, we find that the image of p with respect to this side is

$$x = t_1 + t_2 - t_1t_2\bar{p}.$$

The foot of the perpendicular from p on this side is then equal to half the sum of the point and its image, or

$$x_3 = \tfrac{1}{2}(p + t_1 + t_2 - t_1t_2\bar{p}).$$

The directed length of this perpendicular from p is given by

$$x_3 - p = \tfrac{1}{2}(-p + t_1 + t_2 - t_1t_2\bar{p}).$$

Similarly we have

$$x_2 - p = \tfrac{1}{2}(-p + t_1 + t_3 - t_1t_3\bar{p}),$$
$$x_1 - p = \tfrac{1}{2}(-p + t_2 + t_3 - t_2t_3\bar{p}).$$

We see from these equations that the substitution of p, a point not on any side of the triangle, in the equations of the sides gives twice the directed lengths of the perpendiculars on the sides. If we take the sum of these three vectors and equate it to zero, we see that we have the same expression for p as was given for x by addition of the equations of the sides. This shows that the resultant of two of these vectors is equal in length to the third but in opposite direction. In other words, the sum of the three vectors gives a closed triangle returning to the starting point.

The symmedian point is the only point in the triangle that possesses this property with respect to the sides. This property is at once evident from the well-known fact that the symmedian point is the median point of its own pedal triangle. The feet of the perpendiculars from K upon the sides are

$$x_1 = \tfrac{1}{2}(K + S_1 - t_1 - t_2t_3\overline{K}),$$
$$x_2 = \tfrac{1}{2}(K + S_1 - t_2 - t_1t_3\overline{K}),$$
$$x_3 = \tfrac{1}{2}(K + S_1 - t_3 - t_1t_2\overline{K}).$$

The median point is equal to 1/3 the sum of the vertices, or

$$\frac{x_1 + x_2 + x_3}{3} = \frac{1}{6}(3K + 2S_1 - S_2\overline{K}) = K,$$

since $2S_1 - S_2\overline{K} = 3K$, the known relation between the K's and S's of the triangle and a relation that can easily be verified by substitution of the values of K and $\overline{K}$ in terms of the S's. Since the median point is located at 2/3 of the distance from the vertex to the opposite side on the median and since the distance from the median point to the side is equal to half the resultant of the vectors from the median point to the other two vertices, it follows that the vector sum of the three is zero, but these vectors in the pedal triangle of K are the perpendiculars from K upon the sides of the triangle.

When one is dealing with the incenters it is better to make use of the coordinates described by H. A. DoBell in the February, 1932 number of this MONTHLY. In this system we have

$$t_1 = \frac{T_2T_3}{T_1}, \quad t_2 = \frac{T_1T_3}{T_2}, \quad \text{and} \quad t_3 = \frac{T_1T_2}{T_3}.$$

The T's are roots of the equation

$$T^3 - \Sigma_1T^2 + \Sigma_2T - \Sigma_3 = 0.$$

The incenter then becomes

$$I = -\Sigma_1.$$

The relations between the sigmas and the S's then become

$$S_1 = \frac{\Sigma_2^2 - 2\Sigma_1\Sigma_3}{\Sigma_3},$$

$$S_2 = \Sigma_1^2 - 2\Sigma_2,$$

$$S_3 = \Sigma_3.$$

It is known that the incenter is the Nagel point of the pedal triangle of the circumcenter. The circumcircle of this pedal triangle is the nine point circle, or

$$x = \frac{S_1 - t}{2}.$$

To determine the Nagel point of the base circle we must employ the transformation

$$x' = S_1 - 2x = \frac{\Sigma_2^2 - 2\Sigma_1\Sigma_3}{\Sigma_3} - 2x$$

in which x now becomes $-\Sigma_1$

$$x' = \frac{\Sigma_2^2 - 2\Sigma_1\Sigma_3}{\Sigma_3} + 2\Sigma_1 = \frac{\Sigma_2^2}{\Sigma_3}.$$

The Nagel point of the base circle is thus shown to be equal to Σ_2^2/Σ_3 and this value is easily derived from other geometrical relations in the triangle. The Nagel point is the point of intersection of the lines from the vertices to the points where the opposite sides are tangent to their respective escribed circles. All of the other special points of the triangle can be determined in terms of the complex coordinates by proper manipulations or transformations.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 606. *Proposed by H. T. R. Aude, Colgate University*

An enemy ship A is proceeding on a course directly east at its top speed of 25 miles per hour when it is spied by a more powerful ship B of a speed of only 20 miles per hour. At the instant of discovery the ship B is 10 miles directly south of A. On account of the shore line the enemy ship A must continue its course, while the course of B is altered to $N\theta E$ to bring A within range. If the effective range is not more than 7 miles, find the course of B that will bring about the least distance AB, and the interval of time (while following this course) during which A will be within range.

E 607. *Proposed by V. Thébault, San Sebastián, Spain*

Consider an orthocentric tetrahedron $ABCD$, of orthocenter H. Let O, A', B', C', D' be the circumcenters of the tetrahedra $ABCD$, $BCDH$, $CDAH$, $DABH$, $ABCH$. Prove that the tetrahedra $ABCD$ and $A'B'C'D'$ are homothetic from a center which divides OH in the ratio 3:2. Show also that the lines AA', BB', CC', DD' pass through the centers of gravity of the respective tetrahedra $BCDH$, $CDAH$, $DABH$, $ABCH$.

E 608. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

A certain three-digit number yields a quotient of 26 when divided by the sum of its digits. If the digits are reversed, the quotient is 48. What is the smallest three-digit number for which this is possible?

E 609. *Proposed by Frank Hawthorne, Allegheny College*

Show that the diagonals of three faces of a parallelepiped, drawn from the same vertex and prolonged half their length, determine three points which are coplanar with the opposite vertex.

E 610. *Proposed by Howard Eves, Syracuse University*

(*a*) Show that all closed curves of the same constant diameter, d, have the same perimeter, πd.

(*b*) What is the least area that a closed curve of constant diameter d may have?

(Such a "curve of constant width" touches two parallel lines, distant d apart, drawn in any direction. See, *e.g.*, H. Steinhaus, *Mathematical Snapshots*, 1938, p. 51.)

SOLUTIONS

Conflicting Claims

E 566 [1943, 254]. *Proposed by Michael Wilensky, Cincinnati, Ohio*

Suppose there is a principle according to which a claimant gets a share of the contestable thing, proportionate to his claim; so that when one of two claimants claims the whole, while the other claims only half of it, the former gets three quarters (viz., the incontestable half, and half the contestable half) and the latter gets one quarter (which is half his claim). Find a formula for the share of each of n claimants, when the kth claimant claims $1/k$ of the entity $(k=1, 2, \cdots, n)$.

Solution by H. D. Larsen, University of New Mexico. Denote the rth claimant by C_r and his proportionate share by S_r. Then since C_n's claim of $1/n$ of the entity is contested by each of the other claimants,

$$S_n = \frac{1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}.$$

Further, that part of the whole claimed by C_r but not by C_{r+1} is equal to

$$\frac{1}{r} - \frac{1}{r+1}.$$

Inasmuch as this part is contested by r persons, C_r's share in it is

$$\frac{1}{r}\left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{r^2(r+1)}.$$

But this is precisely the difference between S_r and S_{r+1}; that is,

$$S_r = S_{r+1} + \frac{1}{r^2(r+1)}.$$

This recursion formula may be used to calculate the different shares, starting with $S_n = 1/n^2$.

Also solved by Frank Hawthorne, Monte Dernham, and the proposer.

Editorial Note. Larsen's recursion formula may be written in the form

$$S_r + \frac{1}{r} - \frac{1}{r^2} = S_{r+1} + \frac{1}{r+1},$$

whence

$$S_r + \frac{1}{r} - \sum_r^n \frac{1}{q^2} = S_{r+1} + \frac{1}{r+1} - \sum_{r+1}^n \frac{1}{q^2} = \cdots = S_n + \frac{1}{n} - \frac{1}{n^2} = \frac{1}{n}$$

and, finally,

$$S_r = \sum_r^n \frac{1}{q^2} - \frac{1}{r} + \frac{1}{n}.$$

A Generalization of E 467

E 568 [1943, 260]. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Lucedale, Miss.*

In a given triangle show that the radical axes of the circumcircle with the respective circles whose diameters are any three concurrent Cevians meet the corresponding sides in three collinear points.

Solution by Howard Eves, Syracuse University. Let ABC be the given triangle, O the circumcenter, R the circumradius, and D the midpoint of the Cevian AA' through A. Through A draw AA'' perpendicular to OD, cutting BC in A''. Then, clearly, AA'' is the radical axis of the circumcircle and the circle on AA' as diameter.

We proceed now by vector analytic geometry. Take the origin at O, and let the respective position vectors of A, B, C, A', B', C', A'', B'', C'' be u, v, w, u', v', w', u'', v'', w''. Then we have

$$u \cdot u = v \cdot v = w \cdot w = R^2,$$
$$v \cdot w = R^2 \cos 2A, \qquad w \cdot u = R^2 \cos 2B, \qquad u \cdot v = R^2 \cos 2C. \tag{1}$$

Also, by Ceva's Theorem, there exist constants k_1, k_2, k_3 such that

$$u' = \frac{k_3 v + k_2 w}{k_3 + k_2}, \qquad v' = \frac{k_1 w + k_3 u}{k_1 + k_3}, \qquad w' = \frac{k_2 u + k_1 v}{k_2 + k_1}.$$

Thus the position vector of D is

$$\frac{(k_3 + k_2)u + k_3 v + k_2 w}{2(k_3 + k_2)};$$

and since AA' is perpendicular to OD,

$$(u'' - u) \cdot \{(k_3 + k_2)u + k_3 v + k_2 w\} = 0. \tag{2}$$

But since A'' is on BC, $u'' = (m_3 v - m_2 w)/(m_3 - m_2)$. Substituting this expression for u'' in (2), we find, by means of the relations (1),

$$\frac{m_2}{m_3} = \frac{k_2(1 - \cos 2A + \cos 2B - \cos 2C)}{k_3(1 - \cos 2A - \cos 2B + \cos 2C)}.$$

We may therefore take

$$m_2 = k_2(1 - \cos 2A + \cos 2B - \cos 2C),$$
$$m_3 = k_3(1 - \cos 2A - \cos 2B + \cos 2C).$$

Similarly, if we set

$$m_1 = k_1(1 + \cos 2A - \cos 2B - \cos 2C),$$

we find $v'' = (m_1w - m_3u)/(m_1 - m_3)$ and $w'' = (m_2u - m_1v)/(m_2 - m_1)$. Hence, by Menelaus' Theorem, A'', B'', C'' are collinear, and the theorem is proved.

Also solved by the proposer.

The Butterfly Theorem

E 571 [1943, 326]. *Proposed by Sol Mitchell, University of Toronto*

Let O be the midpoint of a chord AB of a circle, and let CD, EF be any two other chords through O. Prove synthetically that CE and DF meet AB in points equidistant from O.

I. *Solution by Joseph Rosenbaum, Bloomfield, Conn.* Let $D'F'$ be the image of DF by reflection in the diameter through O, and let G, H be the points where CE, DF meet AB. In the notation of Johnson's "directed angles," taking the positive sense to be such that $\sphericalangle EOB$ is acute, we have

$$\sphericalangle BOF' = \sphericalangle FOA = \sphericalangle EOB = \tfrac{1}{2} \sphericalangle EOF' = \sphericalangle ECF',$$

i.e.,

$$\sphericalangle GOF' = \sphericalangle GCF'.$$

Thus O, G, C, F' are concyclic. Hence

$$\sphericalangle OF'G = \sphericalangle OCG = \sphericalangle DCE = \sphericalangle DFO = \sphericalangle OF'D'.$$

Therefore G lies on $D'F'$, and is the reflected image of H.

II. *Solution by W. E. Buker, Pittsburgh Public Schools.* Let CE and DF intersect AB at G and H. respectively. Through H draw a line parallel to CE, intersecting CD and EF at K and L. By similar triangles.

$$\frac{LH}{EG} = \frac{OH}{OG} = \frac{HK}{GC} \quad \text{and} \quad \frac{LH}{FH} = \frac{DH}{KH}.$$

Therefore

$$\frac{OH^2}{OG^2} = \frac{LH \cdot HK}{EG \cdot GC} = \frac{FH \cdot HD}{EG \cdot GC} = \frac{AH \cdot HB}{AG \cdot GB} = \frac{OA^2 - OH^2}{OB^2 - OG^2} = \frac{OA^2}{OB^2} = 1.$$

III. *Solution by Robert Steinberg, University of Toronto.* As this is essentially a theorem of affine geometry, we can replace the circle by a conic. Let CE and DF meet AB at G and H, and let I be the point at infinity on AB. Now, by Steiner's Theorem we have

$$D(AFCB) \overline{\wedge} E(AFCB).$$

Noting the points at which these two pencils meet the line AB, we have

$$AHOB \overline{\wedge} AOGB \overline{\wedge} BGOA.$$

Since the projectivity $AHOB \overline{\wedge} BGOA$ interchanges A and B, it must be an involution. Since it has one double point, O, it must have another which is the harmonic conjugate of O with respect to A and B, namely I. Since H and G are another pair of this involution, these likewise must be harmonic conjugates with respect to O and I. Hence O is the midpoint of GH.

IV. *Solution by E. P. Starke, Rutgers University.* The proposed theorem need not be restricted to a circle: it is true of any conic. Let CE meet DF in P, and let CF meet DE in Q. Then PQ is the polar of O with respect to the conic. But any line through O intersects the conic and PQ in points which, with O, form a harmonic range; therefore A, O, B, and the intersection I, with PQ, are harmonic. But $OA = OB$: thus I must be infinitely distant, *i.e.*, AB and PQ are parallel. Furthermore, in the complete quadrangle, the lines PF, PO, PC, PQ are a harmonic pencil. It intersects AB in points H, O, G, I, which are thus a harmonic range; but I is at infinity, so $GO = HO$, as required.

V. *Solution by J. H. Butchart, Grinnell College.* This is an example of Desargues' Theorem concerning conics passing through the vertices of a complete quadrangle. O is a double point of the involution determined by $CDEF$ on AB. Since A, B form a pair of corresponding points in this hyperbolic involution, the other double point is at infinity, and the points G, H (where CE, DF meet AB) are likewise equidistant from O.

Also solved by Mannis Charosh (*School Science and Mathematics*, October, 1941, p. 684), Howard Eves (in two ways, like IV and V), James Jenkins (in two ways, like II and IV), R. A. Johnson (*Modern Geometry*, p. 78), L. M. Kelly (like IV), and R. K. Morley (like II).

Consecutive Squares

E 572 [1943, 326]. *Proposed by V. Thébault, San Sebastián, Spain*

What scales of notation admit perfect squares ab, bb, and cd, where a, b, c, d are consecutive integers?

Solution by Free Jamison, U. S. Navy Air Navigation School. For any positive integer a, we may use $4a+3$ as radix. Then

$$ab \text{ is } \qquad a(4a+3) + a + 1 = (2a+1)^2,$$

$$bb \text{ is } (a+1)(4a+3)+a+1=(2a+2)^2,$$

and

$$cd \text{ is } (a+2)(4a+3)+a+3=(2a+3)^2.$$

Also solved by D. H. Browne, W. E. Buker, Walter Penney, E. P. Starke, Jerzy Szmojsz, Maud Willey, and the proposer.

Trilinear and Tetrahedral Polarities

E 573 [1943, 326]. *Proposed by N. A. Court, University of Oklahoma*

Given two (three) vertices of a triangle (tetrahedron), determine the remaining vertex so that a given point and a given line (plane) shall be harmonic for the triangle (tetrahedron).

I. *Solution by J. H. Butchart, Grinnell College.* Let the given side AB of the triangle meet the given line l in X. If X' is the harmonic conjugate of X with respect to A and B, and L is the given pole, let $X'L$ meet l in P. Let the harmonic conjugate of l with respect to PL and PB meet AL in Y'. Then BY' meets PL in the third vertex C. The proof is immediate from the harmonic properties of the figure.

For the three-dimensional problem, let the given plane μ meet the plane ABC in the line p whose trilinear polar with respect to ABC is P. The line PM, where M is the given harmonic point, is one locus of the remaining vertex D. Let AP meet BC in X' and let MX' meet μ in M'. Then the harmonic conjugate of X' with respect to M and M' is a point U' on AD. (See N. A. Court, *Modern Pure Solid Geometry*, p. 233.) Thus D is determined as the point where PM meets AU'.

II. *Solution by James Jenkins, University of Toronto.* Project A and B from the given pole L into F and G on the given line l. Then C is the trilinear pole of AB with respect to LFG.

Analogously, project A, B, C from the given pole M into F, G, H on the given plane μ. Then D is the tetrahedral pole of the plane ABC with respect to the tetrahedron $MFGH$.

Also solved by the proposer, who refers to this MONTHLY [1936, 89].

A Quadrilateral in a Semicircle

E 574 [1943, 326]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

If a quadrilateral with sides a, b, c, x is inscribed in a semicircle of diameter x, show that

$$x^3-(a^2+b^2+c^2)x-2abc=0.$$

Solution by R. A. Johnson, Brooklyn College. The proposition is true, only with the interpretation that the quadrangle is convex. Though it is merely a

restatement of Ptolemy's Theorem, it presents a number of striking facets.

Let a, b, c, x be in order the sides of an inscribed quadrangle, x being a diameter of the circle. Then the diagonals, on the authority of Pythagoras, are $(x^2-a^2)^{1/2}$ and $(x^2-c^2)^{1/2}$, and Ptolemy's Theorem states that

$$(x^2 - a^2)^{1/2}(x^2 - c^2)^{1/2} = bx + ac. \tag{1}$$

Simplifying, and removing the extraneous root $x=0$, we have, as a necessary condition,

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0. \tag{2}$$

But the steps are reversible, since all the symbols represent positive quantities; hence either (1) or (2) is a necessary and sufficient condition for the convex cyclic quadrangle.

If we start with a quadrangle in which a and c cross each other, the right side of (1) becomes $ac-bx$; if b and x cross, so that a and c are on opposite sides of x, it is $bx-ac$. In either case, the simplified equation that replaces (2) is

$$x^3 - (a^2 + b^2 + c^2)x + 2abc = 0. \tag{3}$$

This is the necessary and sufficient condition for the existence of a cross-quadrangle. For given values of a, b, c, the roots of (3) are the negatives of the roots of (2); hence we may interpret a negative root of (2) as indicating the existence of a cross-quadrangle.

The implied problem, to construct a cyclic quadrangle with three given sides a, b, c, so that the fourth side shall be a diameter, is solved by taking as diameter any root of (2). It is easy to see that the three roots are always real, and that one is greater than any of a, b, c; there are two negative roots, between which lie all of $-a, -b, -c$ when these are unequal. Hence for any three given lengths a, b, c there exists one convex quadrangle and one cross-quadrangle. (The numerically smaller negative root yields a real circle, in which however the chords a, b, c cannot be drawn in the real domain; each factor of the left side of (1) is imaginary.) It should be remarked that the order of the sides of the quadrangle is immaterial. If one type of cross-quadrangle can be drawn, so can the other.

In general, the problem requires the solution of an irreducible cubic, and is not possible with ruler and compasses. If two of the given lengths, say a and c, are equal, then $-b$ is a root of (2), and the positive root can be found with ruler and compasses. On the other hand, the related problem, to construct a cyclic quadrangle with four given sides, can always be solved with ruler and compasses. (See Johnson, *Modern Geometry*, p. 83.)

We naturally ask what rational solutions (2) may have. If we start by choosing a, c, x, and work with equation (1), we see that if the left side is rational, b will also be rational. The smallest solution of (2) in positive integers with $a<b<c$ seems to be

$$a = 2, \quad b = 7, \quad c = 11, \quad x = 14.$$

There is another approach to the problem. If we assume it to be known that of all quadrangles with four given sides the cyclic one has maximum area, we may go further and ask how to enclose the greatest area with three given lengths and a fourth line of unspecified length. The formula for the area of a cyclic quadrangle is well known (*op. cit.*, p. 82), *viz.*

$$16F^2 = (-a+b+c+x)(a-b+c+x)(a+b-c+x)(a+b+c-x).$$

If we equate to zero the derivative of this expression with regard to x, we are led to equation (2). Thus we have the required maximum when x is the diameter of the circle, a result known to Legendre (Appendix to Book IV, 1794).

Also solved by F. A. Alfieri, D. H. Browne, J. H. Butchart, A. C. Cohen, Jr., William Douglas, Howard Eves, Charles Hatfield, Jr., Frank Hawthorne, Free Jamison, Elmer Latshaw, E. P. Starke, C. F. Strobel, Jerzy Szmojsz, H. B. Thornton, C. W. Topp, W. Unterberg, Alan Wayne, and Maud Willey.

Harmonic Progression

E 576 [1943, 386]. *Proposed by M. A. Dernham, San Francisco*

What well known type of sequence may be defined as one in which each term but the first may be obtained by taking the preceding term and a common constant and dividing their product by their sum?

Solution by J. H. Butchart, Grinnell College. If $a_{n+1}=ca_n/(c+a_n)$, then

$$\frac{1}{a_{n+1}} = \frac{1}{a_n} + \frac{1}{c}.$$

Hence $\{1/a_n\}$ is an arithmetic progression, and $\{a_n\}$ is a harmonic progression.

Also solved by R. K. Allen, W. E. Buker, Howard Eves, Frank Hawthorne, Irving Kaplansky, O. J. Karst, E. P. Starke, Alan Wayne, and the proposer.

Isosceles Tetrahedra

E 577 [1943, 386]. *Proposed by V. Thébault, San Sebastián, Spain*

Given an "isosceles" tetrahedron $A_1A_2A_3A_4$ (so that every two opposite edges are equal), let perpendiculars be drawn to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ at their circumcenters O_1, O_2, O_3, O_4, to meet the hemispheres described exteriorly (or interiorly) on the respective circumcircles in P_1, P_2, P_3, P_4. Show that the tetrahedra $O_1O_2O_3O_4$ and $P_1P_2P_3P_4$ are isosceles, and that they have the same centroid as $A_1A_2A_3A_4$.

Solution by Howard Eves, Syracuse University. The vertices A_1, A_2, A_3, A_4 are interchanged in pairs by half-turns about the three bimedians. These half-turns also interchange the *faces* in pairs, and operate similarly on any four points homologously related to the faces. In particular, they interchange pairs of O_1, O_2, O_3, O_4, and of P_1, P_2, P_3, P_4. Hence the tetrahedra $O_1O_2O_3O_4$ and $P_1P_2P_3P_4$ are isosceles. Finally, the axes of the half-turns concur at an invariant

point, which is equidistant from all the A's, from all the O's, and from all the P's. This is the common circumcenter, incenter, and centroid of all the tetrahedra.

For an excellent introduction to the isosceles tetrahedron see N. A. Court, *Modern Pure Solid Geometry*, pp. 94–102. In connection with the present problem, cf. Exs. 16 and 17, p. 102.

Also solved by J. H. Butchart and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4108. *Proposed by G. Pólya, Stanford University*

Let ${}_nP_r$ be the number of those permutations of n elements which are the products of exactly r cycles without common elements. For instance, ${}_4P_2=11$. Let ${}_nQ_r$ be the number of different classifications of n distinct elements into exactly r classes. For instance, ${}_4Q_2=7$. Prove that

$$(1) \qquad {}_nP_1x + {}_nP_2x^2 + \cdots + {}_nP_nx^n = x(1+x)(2+x)\cdots(n-1+x),$$

$$(2) \qquad {}_nQ_1x + {}_nQ_2x(x-1) + \cdots + {}_nQ_nx(x-1)\cdots(x-n+1) = x^n.$$

Note. In the Calculus of Finite Differences, ${}_nP_r$ and ${}_nQ_r$ are considered in various notations and are often called factorial coefficients; see, for instance, Steffensen, *Interpolation*, Baltimore, 1927, pp. 53–58.

4109. *Proposed by N. A. Court, University of Oklahoma*

Prove that the sum of the n^2 powers of n given points with respect to the n spheres having for diameters the n segments joining the given points to a variable point in space is constant.

4110. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest integer whose square is the sum of two squares in seven different ways.

SOLUTIONS

Perfect and Nearly Perfect Squares

4055 [1942, 549]. *Proposed by V. Thébault, San Sebastián, Spain*

Find two numbers of ten digits of the form *aabbccddee* such that one is a perfect square and the other is a perfect square increased by 7.

Editorial Note. The proposer gave the following results without indicating his method of deduction:

$$5500002244 = 74162^2, \qquad 7744000000 = 88000^2, \qquad 1199998888 = 34641^2 + 7.$$

In the first part it is easily seen that $N = aabbccddee = 11(a0b0c0d0e) = 11M$, and that $a+b+c+d+e$ is a multiple of 11. The problem of finding the integer z so that $N = z^2$ reduces to finding z so that $M = 11Z^2$. We have $3000 < Z = pqrs < 9100$; and one method of determining all the answers is to find first the digits r, s so that the last two digits of M are Oe, and then, p, q so that M ends in $0d0e$. This may be done by the method of differences as follows; for $Z = rs$, we have

$$11Z^2 = r^2(r^2 + 2rs)(2rs + s^2)s^2$$

$$\Delta_s = 2r(2r + 2s + 1)(2s + 1).$$

We carry out the work at first for the last two digits in the following tabular form and supply later the rest of the table if needed

s, Δ_s				
0 Δ_0			0 $2r+1$	0 1
1 Δ_1			$2r+1$ $2r+3$	1 3
2 Δ_2			$4r+4$ $2r+5$	4 5
3 Δ_3			$6r+9$ $2r+7$	9 7
4		1	$8r+7$ . . .	6 . . .

and this is continued up to $s = 9$ inclusive. The work may be checked by the use of $s = 10$. For $s = 2$, the entry $4r+4$ gives the digit 0 when $r = 4$, 9; and proceeding in this manner we find that the last two digits for a possible answer in the order of getting them are 42, 92, 08, 58. This gives the second answer of the proposer for the first part. We next consider in turn $pq42$, $pq92$, $pq08$, $pq58$. In the work for these we omit the known last two digits and the first part of the table which may be supplied later when needed. We proceed as follows for $pq42$

0 Δ_0			$9p$ $2p+2$	$2p+1$ 0	$4p+9$ 2	4 4		
1 Δ_1			$p+2$ $2p+4$	$2p+2$ 2	$4p+1$ 2	8 4		
2 Δ_2			$3p+6$ $2p+6$	$2p+4$ 4	$4p+4$ 2	2 4		
3			$5p+2$ . . .	$2p+8$ . . .	$4p+6$ . . .	6 . . .		

For $q=2$ we have as possible answers $p=4$, 9; and finally we have 4282, 3842, 6742, 8842 after rejection of those too large or too small. We now reexamine this last table with respect to the two next digits to the left, and we find that all but 6742 fail to give a third 0 in the desired place. On completing the necessary part of the table we find that 6742 gives the desired result. The remaining three sets are found to fail in the same way as above except 08 which gives the answer already found. Thus there are only two answers to the first part.

In the second part where $N=z^2+7$, we must have $z\equiv\pm 2 \bmod 11$. We may proceed in a similar manner using differences to find the last two digits of z, then two more digits and finally a fifth digit. The last two digits of z must be among 41, 91, 02, 52, 48, 98, 09, 59. For 41 the values of pq are 12, 62, 33, 83, 04, 54, 25, 75, 46, 96, Denoting the first digit of z by n, we use the above congruence to find trial values of n, in most cases two values of n are found. Then, since $z\geqq 33160$, four of the resulting 16 values are discarded, and then from an estimate of the first two digits of the trial z^2 all the others except 34641 are discarded, and this latter is found to be a solution. The next six sets of last two digits rs do not yield solutions, failing mostly in an estimate of the first two digits for aa. Finally for $rs=59$ there are thirteen cases all of the first twelve failing as above, whereas the last and largest gives a solution

$$99559^2 + 7 = 9911994488.$$

Discarding the cases $a=0$, there are only two solutions.

Line of Images as Axis of Perspectivity

4056 [1942, 616]. *Proposed by J. R. Musselman, Western Reserve University*

Let the line of images of any point T on the circumscribe of triangle $A_1A_2A_3$ cut the sides A_jA_k in the points A_i'. The perpendiculars to the sides A_jA_k at A_i' form the triangle $B_1B_2B_3$; show that the straight lines A_iB_i meet in T.

Solution by Howard Eves, Syracuse University. Let A_i''' be the foot of the perpendicular from T on A_jA_k. Then, since $A_j'''TA_k'''A_i$ is similar to $A_j'B_iA_k'A_i$, it follows that T, B_i, A_i are collinear. Hence the theorem.

Note that $A_1'A_2'A_3'$ need not be the line of images for T, but any line parallel to the Simson line of T with respect to $A_1A_2A_3$.

Editorial Note. The line of images of T is the directrix of a parabola with the focus T and tangent to the sides of $A_1A_2A_3$; hence the parabola is tangent also to the sides of $B_1B_2B_3$. Thus the two triangles have with respect to T the same Simson line and the same line of images. If instead of the line of images we use any arbitrarily chosen parallel, as in the above note, the circumcircle of $B_1B_2B_3$ passes through T, and the line of images of the two triangles are parallel, and by the theorem of the note the point T and the chosen parallel are respectively the center and axis of perspectivity of the two triangles.

Euler's Constant

4046 [1942, 479]. *Proposed by Otto Dunkel, Washington University*

Show that γ, Euler's constant, is given by

$$\gamma = 2\left[1 - \log 2 - \frac{\tau_3}{3} - \frac{\tau_5}{5} - \cdots\right], \qquad \tau_r = \sum_{i=1}^{\infty} \frac{1}{(2i+1)^r}.$$

Solution by the Proposer. We prove first that

$$\sum_{i=1}^{\infty} \frac{1}{i(2i+1)} = 2(1 - \log 2). \tag{1}$$

We have

$$\sum_{i=1}^{n} \frac{1}{i(2i+1)} = 2\left[\sum_{i=1}^{n} \frac{1}{2i} - \sum_{i=1}^{n} \frac{1}{2i+1}\right] = 2\left[1 - \sum_{i=1}^{2n+1} (-1)^{i-1} \frac{1}{i}\right],$$

and, if we let $n \to \infty$, we have (1). A simple modification of the development

$$\log\left(\frac{n+1}{n}\right) = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]$$

gives

$$-\left[\frac{1}{n} - \log\left(\frac{n+1}{n}\right) - \frac{1}{n(2n+1)}\right] = 2\sum_{i=1}^{\infty} \frac{1}{(2i+1)(2n+1)^{2i+1}}. \tag{2}$$

The sum of the left member of (2) from $n=1$ to $n=\infty$ is $-\gamma+2(1-\log 2)$, whereas the summation on the right gives a double series, obviously convergent. Summation by columns gives the equality of the problem.

In order to make the proof complete we shall show the existence of the number γ. From the graph of $1/(1+x)$ from $x=0$ to x, we see that $\log(1+x)$

lies between the areas of two trapezoids, the larger formed by the ordinates 1 and $1/(1+x)$ while the smaller has parallel sides cut off by the tangent at $x/2$. After a simple modification of the two trapezoid areas this gives

$$(3) \qquad x - \frac{x^2}{2(1+x/2)} < \log(1+x) < x - \frac{x^2}{2(1+x)},$$

$$(4) \qquad \log(1+x) = x - \frac{x^2}{2(1+\theta x)}, \qquad \frac{1}{2} < \theta < 1,$$

$$(5) \qquad C(n) = \sum_{i=1}^{n} \frac{1}{i} - \log(n+1) = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i^2}\left(\frac{1}{1+\theta_i/i}\right) < \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i^2} < \frac{\pi^2}{12}.$$

Thus $C(n)$ increases with n and approaches a number $\gamma < \pi^2/12$. We may write

$$\frac{x^2}{2(1+x)} < x - \log(x+1) < \frac{x^2}{2(1+x/2)},$$

$$\frac{1}{2}\left[1 - \frac{1}{n+1}\right] < C(n) < \sum_{i=1}^{n} \frac{1}{i(2i+1)},$$

$$.5 < \gamma < 2(1 - \log 2) = .614^{-}.$$

Replacing i and n by $i+a$ and $n+a$, where a is a fixed positive constant, we can consider similar expressions $C(n, a)$ which are useful in the study of more complex infinite series of the type (1).

Editorial Note. Since the above formula is not convenient for computation, we shall give a more suitable expression for limits l similar to γ, where l is defined in the following theorem:

Given the real function $f(x)$ of the real variable x which has finite derivatives $f'(x), f''(x), \cdots, f^{\text{vi}}(x)$ for all values of $x \geqq 1$; $f^{\text{vi}}(x)$ is continuous and has always the same sign for such values; and $|f^{\text{vi}}(x)|$ decreases as x increases and approaches zero; then $F(n)$, where n is a positive integer, and

$$(1)\, F(n) = \sum_{j=1}^{n} f'(j) - \left[f(n+1) - \frac{1}{2}f'(n+1) + \frac{1}{12}f''(n+1) - \frac{1}{6!}f^{\text{iv}}(n+1)\right]$$

has a finite limit l as the integer $n \to \infty$.

In order to show this we carry out five integrations by parts to obtain

$$(2) \qquad f'(x) = \Delta f(x) - \frac{1}{2}\Delta f'(x) + \frac{1}{12}\Delta f''(x) - \frac{1}{4!}\int_0^1 \psi_4(t) f^{\text{v}}(x+t)\,dt,$$

where $\psi_4(t) = t^2(t-1)^2$, $\Delta f^{(i)}(x) = f^{(i)}(x+1) - f^{(i)}(x)$, and setting $f'(x+1) + f'(x) = \Delta f'(x) + 2f'(x)$.

We now extend (2) by the use of the two integrations

$$\int_0^1 \psi_4(t) f^{\text{v}}(x+t)\,dt = \frac{1}{30} f^{\text{v}}(x+1) - \frac{1}{30}\int_0^1 [6t^5 - 15t^4 + 10t^3] f^{\text{vi}}(x+t)\,dt,$$

$$\int_0^1 tf^{\text{vi}}(x+t)dt = f^{\text{v}}(x+1) - \Delta f^{\text{iv}}(x);$$

and we have

$$f'(x) = \Delta f(x) - \frac{1}{2}\Delta f'(x) + \frac{1}{12}\Delta f''(x) - \frac{1}{6!}\Delta f^{\text{iv}}(x) + \frac{1}{6!}\int_0^1 \psi_6(t) f^{\text{vi}}(x+t)dt,$$

(3)
$$\psi_6(t) = 6t^5 - 15t^4 + 10t^3 - t = t(t-1)(2t-1)(3t^2-3t-1),$$

where $\psi_6(t) = -\psi_6(1-t)$, $\psi_6(t) < 0$, for $0 < t < 1/2$ and $\psi_6(t) > 0$ for $1/2 < t < 1$. Summing each member of (3) for $x = 1, 2, \cdots, n$, we have finally

$$F(n) = -\left[f(1) - \frac{1}{2}f'(1) + \frac{1}{12}f''(1) - \frac{1}{6!}f^{\text{iv}}(1)\right] + \frac{1}{6!}\sum_{j=1}^{n}\int_0^1 \psi_6(t) f^{\text{vi}}(j+t)dt. \tag{4}$$

The integral on the right is the sum of two integrals, the first from 0 to 1/2 and the second from 1/2 to 1, and we have the sum of $2n$ terms with alternating signs; and it is easily seen from the graph that the absolute values of the terms decrease. As n increases, the absolute value of the sum of the $2n$ terms increases, and hence $F(n)$ increases, or decreases, according to the sign of $f^{\text{vi}}(x)$; also for $n \to \infty$, the infinite series is convergent since the terms approach zero, and $F(n)$ approaches a finite limit l

$$l = -\left[f(1) - \frac{1}{2}f'(1) + \frac{1}{12}f''(1) - \frac{1}{6!}f^{\text{iv}}(1)\right] + \frac{1}{6!}\sum_{j=1}^{\infty}\int_0^1 \psi_6(t) f^{\text{vi}}(j+t)dt, \tag{5}$$

$$l = F(n) + \frac{1}{6!}\sum_{j=n+1}^{\infty}\int_0^1 \psi_6(t) f^{\text{vi}}(j+t)dt. \tag{6}$$

The absolute value of the last term on the right of (6) is less than $|f^{\text{vi}}(n+1)|/5!2^7$; and (6) gives an expression for approximating l. Thus for $f(x) = \log x$ we have

$$\gamma = \sum_{j=1}^{n}\frac{1}{j} - \left[\log(n+1) - \frac{1}{2(n+1)} - \frac{1}{12(n+1)^2} + \frac{1}{5!(n+1)^4}\right] - \frac{1}{5!}\sum_{j=n+1}^{\infty}\int_0^1 \frac{\psi_6(t)}{(j+t)^6}dt, \tag{7}$$

with the last term less than $1/2^7(n+1)^6$. For $n = 9$ the last term is less than .00000000782, and an easy computation of the remaining part gives the value of γ correct in the first seven decimals.

It is easily seen how to extend all of the above to derivatives of higher order than six, and the rule for obtaining the polynomials $\psi_k(t)$ will be discovered to be as follows. We integrate $\psi_5(t)$ from $t = 0$ to t and the result may be taken for $\psi_6(t)$; in the above we have discarded a least common integral denominator

simply for convenience. We then proceed similarly with $\psi_6(t)$ to find $\psi_7(t)$ but here we add to the integral a term of the first degree ct and determine the coefficient c so that the sum is zero for $t=1$. We continue in this way as far as we please, distinguishing the cases $\psi_{2m}(t)$ and $\psi_{2m+1}(t)$. It will be seen that

$$\psi_k(t) = A_k t^k - A_{k-1}t^{k-1} + A_{k-2}t^{k-2} - A_{k-4}t^{k-4} + A_{k-6}t^{k-6} - \cdots, \quad k \geqq 4,$$

where the coefficients A_{k-i} are positive; and that $\psi_{2m}(t) = \psi_{2m}(1-t)$, $\psi_{2m+1}(t) = -\psi_{2m+1}(1-t)$. From the above integration rule we infer that $\psi_k(t)/kA_k$ for t a positive integer is the sum $1+2^{k-1}+3^{k-1}+\cdots+(t-1)^{k-1}$. If we set $f(x) = x^k$, $k=1, 2, \cdots, 6$, in (5) and (6), we obtain these latter polynomials.

There are other important applications of (5) and (6); for example if we set $f(x) = -1/2x^2$, we obtain an approximate value of the infinite series $1+1/2^3+1/3^3+\cdots$, and if we take $n=9$, the result is correct in the first eight decimals using the estimate of error under (6). This gives $\tau_3 = .05179978$. The following evaluation, correct in the given figures, involves laborious computations

$$\sum_{i=1}^{\infty} \frac{1}{i^{1.1}} = 10.58444846 \cdots .$$

Triangles Having Integral Sides and Areas

4047 [1942, 479]. *Proposed by T. R. Running, Ann Arbor, Mich.*

Triangles have the sides $x-1$, x, $x+1$, the altitude h with x as base, and area A, where x, h, A are whole numbers. The first six possible triangles are given by the table

n	h	x	A
0	0	2	0
1	3	4	6
2	12	14	84
3	45	52	1170
4	168	194	16296
5	627	724	226974
6	.	.	.

Do the relations

$$h_{n+2} = 4h_{n+1} - h_n, \qquad x_{n+2} = 4x_{n+1} - x_n, \qquad A_{n+2} = 14A_{n+1} - A_n,$$

hold for all the triangles fulfilling the given conditions?

Note by W. B. Clarke, San Jose, Calif. In the *National Mathematics Magazine*, Nov. 1934, p. 63, a solution was given by E. P. Starke of my problem No. 65 which stated:

Considering only triangles whose sides are consecutive integers, and whose area is an integer, find the area of the triangle which is next larger than the one whose area is 16296.

In a footnote to his solution Starke referred to the solution of 3677 in this

MONTHLY which would appear later, 1935, 572. This has a direct bearing on 4047; and from these developments it appears to me that the answer to 4047 is yes.

Solution by E. P. Starke, Rutgers University. The proposed question has an affirmative answer. By familiar formula

$$A = \tfrac{1}{2}hx = \sqrt{\tfrac{1}{2}(3x)\cdot\tfrac{1}{2}(x+2)\cdot\tfrac{1}{2}x\cdot\tfrac{1}{2}(x-2)} = \tfrac{1}{4}x\sqrt{3x^2-12}.$$

Since A is integral, x must be even. Put $x=2y$ to obtain

$$A = hy, \qquad h = \sqrt{3(y^2-1)}.$$

Thus h must be a multiple of 3, and we may put $h=3z$ to obtain $A=3yz$ and

$$y^2 - 3z^2 = 1. \tag{1}$$

The problem is thus equivalent to that of finding all pairs of positive integers (y, z) which satisfy (1). Consider

$$y_{n+1} = 2y_n + 3z_n, \qquad z_{n+1} = 2z_n + y_n, \tag{2}$$

or, solved for y_n, z_n,

$$y_n = 2y_{n+1} - 3z_{n+1}, \qquad z_n = 2z_{n+1} - y_{n+1}. \tag{3}$$

If (y_n, z_n) satisfy (1), it is easily verified that (y_{n+1}, z_{n+1}) do also, and conversely. Also (2) implies the equations

$$y_{n+2} = 2y_{n+1} + 3z_{n+1}, \qquad z_{n+2} = 2z_{n+1} + y_{n+1}.$$

If the three z's and the three y's, respectively, are eliminated between these equations and (2) we have

$$y_{n+2} = 4y_{n+1} - y_n, \qquad z_{n+2} = 4z_{n+1} - z_n. \tag{4}$$

If y_n, z_n are non-negative, (2) implies $y_{n+1}>y_n$, $z_{n+1}>z_n$. Further if y_{n+1}, z_{n+1} are positive integers satisfying (1), $9z_{n+1}^2 = 3y_{n+1}^2 - 3 < 4y_{n+1}^2$ implies $3z_{n+1} < 2y_{n+1}$ and y_n is positive by (3). Similarly $y_{n+1}^2 = 3z_{n+1}^2 + 1 < 4z_{n+1}^2$ unless $z_{n+1}=1$ implies $y_{n+1} < 2z_{n+1}$ and z_n is positive. Thus if (y_{n+1}, z_{n+1}) is any solution of (1) and $z_{n+1} \neq 1$, there is obtained from (3) a solution (y_n, z_n) in smaller positive integers, and then from (y_n, z_n) another still smaller, *etc.* Since there cannot be an endless series of positive integral values of z each less than the preceding, the above reduction process must stop somewhere: *i.e.* at some step the value of z becomes 1. By now reversing the process we see that any solution of (1) is obtained from $y_0=1, z_0=0$; $y_1=2$, $z_1=1$ by repeated use of (2).

Multiplying the equations (4) by 2 and 3 respectively we have the proposed equations in x and h, valid for all triangles fulfilling the given conditions. The identity

$$2y_{n+1}z_{n+1} = (2y_{n+1} - 3z_{n+1})(y_{n+1} + 2z_{n+1}) + (2z_{n+1} - y_{n+1})(2y_{n+1} + 3z_{n+1})$$

by use of (2) and (3) becomes

$$2y_{n+1}z_{n+1} = y_nz_{n+2} + y_{n+2}z_n. \tag{5}$$

From (4) we may put

$$(4y_{n+1})(4z_{n+1}) = (y_{n+2} + y_n)(z_{n+2} + z_n) = y_{n+2}z_{n+2} + y_nz_{n+2} + y_{n+2}z_n + y_nz_n,$$

which can be simplified by (5) to the form

$$14y_{n+1}z_{n+1} = y_nz_n + y_{n+2}z_{n+2}.$$

Finally, since $3yz = A$, we have, upon multiplying by 3, the third of the proposed relations, valid also for all triangles fulfilling the given conditions.

Triangle Variable in Space

4061 [1942, 688]. *Proposed by N. A. Court, University of Oklahoma*

A variable triangle has its vertices on three skew straight lines, and two of its sides meet a given plane in points lying respectively on two straight lines. Show that the points of intersection of that plane with the third side of the variable triangle are collinear.

Solution by L. S. Sinclair, University of Toronto. Let the three skew lines be l, m, n, meeting the given plane in L, M, N; and let p, r be the two lines given in that plane. Let the variable triangle be ABC, with A on l, B on m, C on n, AB intersecting r, and BC intersecting p. It is required to prove that the locus of the point where AC meets the plane pr is a line.

The variable line AB, intersecting the three skew lines l, m, r, sweeps out a quadric; so also does BC (which intersects m, n, p). Hence the points A, B, C vary on l, m, n, in such a way that

$$A \barwedge B \barwedge C.$$

Therefore AC joins corresponding points of projective ranges on two skew lines, and sweeps out a third quadric. Since the triangle LMN is one possible position for ABC, the side LN is one generator of this last quadric; *i.e.*, LN forms part of the section of the quadric by the plane pr. Hence the rest of the section is another line, which is the desired locus.

Solved also by Paul Brock, W. W. Dolan and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Sister Helen Sullivan of Mount St. Scholastica College was recently elected National Historian of Kappa Mu Epsilon.

The mathematics department of University of Cincinnati announces the following changes: Assistant Professors E. T. Miller and H. A. Sibert hold the

rank of major in the army; J. L. Baker and Assistant Professor E. F. White are lieutenants in the army and navy respectively; Dr. Gabriel Horvay is on leave of absence for war research. Associate Professor H. L. Miller has been promoted to a professorship; Assistant Professors H. A. Dangel and A. H. Knebel have been promoted to associate professorships; Dr. C. A. Ludeke, W. E. Restemeyer and E. T. Miller have been promoted to assistant professorships; Professor E. S. Smith (Major, U.S.A.) has retired from the army to resume academic status.

Dr. Claire F. Adler has been appointed to an assistant professorship at New York University.

Assistant Professor P. O. Bell of the University of Kansas has been promoted to an associate professorship.

Associate Professor Ethel B. Callahan of Shepherd State Teachers College, Shepherdstown, West Virginia, has been appointed to an associate professorship at Hartwick College, Oneonta, New York.

Sister James S. Creane has been appointed to an assistant professorship at Fontbonne College, St. Louis, Missouri.

Assistant Professor Paul Eberhart of Washburn University, Topeka, Kansas, has been promoted to a professorship.

W. E. Ekman of the University of South Dakota has been promoted to a professorship of mathematics and astronomy.

Assistant Professor H. F. Fehr of the State Teachers College at Upper Montclair, New Jersey, has been promoted to an associate professorship.

Edna M. Feltges of Woodrow Wilson Junior College, Chicago, has been appointed chairman of the department of mathematics.

M. P. Fobes of the College of Wooster, Ohio, has been promoted to an assistant professorship.

R. M. Foster of Bell Telephone Laboratories has been appointed head of the department of mathematics of the Polytechnic Institute of Brooklyn.

Assistant Professor J. W. Foust of Central Michigan College of Education has been promoted to a professorship.

W. A. Gager of the Junior College in St. Petersburg, Florida, has been appointed to an associate professorship at the University of Florida.

J. B. Greeley of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at Lafayette College.

Assistant Professor E. H. Hadlock of Hastings College, Nebraska, has been promoted to an associate professorship.

Dr. M. C. Hartley has been appointed to an assistant professorship at the University of Illinois.

Associate Professor H. H. Hartzler of Goshen College, Indiana, has been promoted to a professorship.

Assistant Professor P. R. Hill of the University of Georgia has been promoted to an associate professorship.

Assistant Professor C. M. Howard of North Texas Agricultural College has been promoted to a professorship and is acting head of the department.

Dr. P. M. Hummel of the University of Alabama has been promoted to an assistant professorship.

Associate Professor J. W. Hurst of Montana State College has been promoted to a professorship.

Associate Professor J. A. Hyden of Vanderbilt University has been promoted to a professorship.

Assistant Professor F. B. Jones of the University of Texas has been promoted to an associate professorship. He is now on leave, serving in the Underwater Sound Laboratory at Harvard University.

Dr. H. T. Karnes of Louisiana State University has been promoted to an assistant professorship.

Dr. Fulton Koehler of the Northwest National Life Insurance Company has been appointed to an assistant professorship at the University of Minnesota.

J. E. LaFon of the University of Oklahoma has been promoted to an assistant professorship.

Assistant Professor R. G. Lubben of the University of Texas has been promoted to an associate professorship.

Assistant Professor W. T. MacCreadie of Bucknell University has been promoted to an associate professorship.

Associate Professor F. L. Manning of Ursinus College has been promoted to a professorship.

J. N. McClelland of Drake University has been promoted to an assistant professorship.

Associate Professor C. E. Melville of Clark University has been promoted to a professorship.

Professor H. A. Meyer of Hanover College is acting professor of mathematics at Indiana University.

Assistant Professor R. R. Middlemiss of Washington University has been promoted to an associate professorship.

Associate Professor C. G. Phipps of the University of Florida has been promoted to an associate professorship.

R. J. Pitts of Fort Valley State College, Georgia, has been promoted to an assistant professorship.

Associate Professor W. G. Pollard of the University of Tennessee has been promoted to a professorship.

Professor G. C. Priester of the Institute of Technology at the University of Minnesota has been made head of the department of mathematics and mechanics.

Dr. Adrienne S. Rayl of the University of Alabama, Birmingham Center, has been promoted to an assistant professorship.

Professor O. H. Rechard of the University of Wyoming has been relieved of the chairmanship of the department of mathematics so that he may serve as War Projects Administrator. Associate Professor C. F. Barr has been appointed acting chairman of the department.

Assistant Professor P. K. Rees of Southern Methodist University has been appointed to a professorship at Southwestern Louisiana Institute.

Dr. C. E. Rhodes of Union College has been promoted to an assistant professorship.

H. L. Rice of the University of Omaha has been promoted to an assistant professorship.

Assistant Professor F. A. Rickey of Louisiana State University has been promoted to an associate professorship.

Assistant Professor Robin Robinson of Dartmouth College has been promoted to a professorship and appointed chairman of the department of mathematics and astronomy.

Assistant Professor W. J. Robinson of Marshall College, Huntington, West Virginia, has been appointed to an assistant professorship at Allegheny College.

Assistant Professor E. B. Roessler of the University of California College of Agriculture has been promoted to an associate professorship.

A. L. Starrett of Georgia School of Technology has been promoted to an assistant professorship.

Dr. R. E. Street has been appointed to a visiting lectureship in physics at Dartmouth College.

I. L. Stright has been appointed to an assistant professorship at Baldwin-Wallace College.

Associate Professor W. R. Talbot of Lincoln University, Jefferson City, Missouri, has been promoted to a professorship.

J. E. Thompson of Pratt Institute, Brooklyn, has been promoted to an associate professorship.

Dr. H. C. Trimble of Iowa State Teachers College has been promoted to an assistant professorship.

Assistant Professor P. L. Trump of the University of Wisconsin has been promoted to an associate professorship in the teaching of mathematics.

R. N. Van Arnam of Lehigh University has been promoted to an assistant professorship of mathematics and astronomy.

Professor C. C. Wagner of Pennsylvania State College has been appointed assistant dean of the School of Liberal Arts.

Assistant Professor D. L. Webb of Texas Technological College has been promoted to an associate professorship.

Associate Professor P. D. Wilkins of Bates College has been promoted to a professorship.

Assistant Professor C. R. Wylie, Jr., of Ohio State University has been given leave of absence for service at Wright Field, Dayton, Ohio.

The following appointments to instructorships are announced:
Antioch College: C. B. Helms
Brooklyn College: Solomon Hurwitz, Joseph Landin
Brown University: W. M. Kincaid
Duke University: Dr. L. I. Wade
Michigan State College: Nicholas Musselman
State College of Washington: Dr. P. F. Nemenyi
University of Arizona: Dr. O. B. Ader
University of Cincinnati: Robert Fopma
University of Illinois: Dr. Theodore Bedrick
University of Nebraska: H. W. Linscheid
University of Rochester: Dr. Dorothy L. Bernstein

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

NOTES ON THE NAVY V-12 PROGRAM

Qualifications of enlisted men now entering the V-12 Program. "In order to qualify for the program, an enlisted man already in active service must have had a minimum of two years of high school mathematics in order to qualify. Civilians qualify through a written examination, and the most recent examination given, namely, the one on 9 November 1943, was quite heavily weighted in mathematics."—Commander Alvin C. Eurich, Officer in Charge, Standards and Curriculum Section.

Teaching materials. "The Navy has purposefully refrained from making specific recommendations concerning text books to be used. It is felt that that is a matter which can be left to the institutions. The Navy feels that it has selected qualified colleges and universities, and that these courses are very similar to those which they have been accustomed to giving, and, therefore, they should be expected to rely upon their own experience for determining the materials which can be used most effectively."—Commander Alvin C. Eurich.

Faculty teaching loads. "The reference in Bulletin No. 101 to faculty teaching load, refers primarily to the amount of teaching which a faculty member does during a particular term. There have been instances where faculty members apparently have felt that, although student programs should be accelerated and everybody else in the country should work harder, their teaching load should remain at a normal level, which in some instances was very low, and was designed to give the faculty member considerable time for research. The Navy is contracting in the V-12 program for teaching only, and not for research—important though the research may be. Summer teaching is taken care of by allowance of proportionate extra compensation based on the regular salary of the instructor. The Navy does not stipulate that instructors have to teach during the summer; however, with the instructional shortage which there seems to be in certain fields, it is believed that most institutions would be faced with a very serious problem if faculty members were to teach the customary eight or nine months only."—Commander Alvin C. Eurich.

General engineering quotas. "It is . . . announced that present Navy V-12 upper-level students in the field of General Engineering are adequate to meet current needs of the Navy. Therefore, no students shall be recommended for assignment to this specialty at this time."—Navy V-12 Bulletin No. 155.

THE TRAINING PROGRAMS IN METEOROLOGY

In view of the large number of inquiries relative to the discontinuance of the

meteorological training programs sponsored by the Army, the following statement was solicited from Professor C. G. Rossby, University of Chicago, Chairman of the University Meteorological Committee.

"The Office of the Assistant Chief of Air Staff, Operations, Commitments and Requirements, determines from information submitted to them the number of personnel required for a particular branch of the Army Air Forces. The University Meteorological Committee, which is composed of representatives of the five universities training meteorologists at the professional level and the Army Training School at Chanute Field, Illinois, was informed by the Assistant Chief of Staff that the requirements of the Air Forces for meteorologists through the year 1944 had been met and that therefore there would not be any additional meteorologists trained unless the requirements for 1945 and 1946 indicated a need for additional meteorologists.

"The B and C programs were planned to prepare students for entrance into the meteorological programs conducted by the A schools. The curricula stressed mathematics and physics particularly.

"As the training programs in the A schools were discontinued for the present, there was not any reason for having additional B and C programs. However, the value of these preparatory training programs was considered so great by the Army Air Forces that it was decided to allow those students who were already enrolled in the B and C programs to complete their courses.

"A reclassification board from the Flying Training Command visited the schools having B and C programs and offered the students the following choices of additional training programs at the completion of their B and C courses: flight training including pilot, navigator, and bombardier; communications training; weather observing; advanced engineering training and other Army Specialized Training Program subjects.

"The men were allowed to indicate their choices in the order of their preference. At the completion of the present courses they will be assigned to one of the above enumerated programs.

"The above statement of the policy of the Army Air Forces prohibits us from accepting any further applications for the A, B, or C programs at the present time."

THE ALIEN BOOK REPUBLICATION PROGRAM

Approximately a year ago, the Office of Alien Property Custodian, Washington, D. C., announced to scientists, libraries, industrial and research organizations that it was considering a Book Republication Program in order to make certain foreign scientific materials available for use in the war effort. A thirty-nine page booklet was distributed at that time, containing titles for which one or more republication requests had already been received. Those interested in securing copies of any of the titles there listed were requested to advise the Custodian. As a result of the investigation then made, and currently continuing, licenses have been granted to certain individuals and publishers for the repro-

duction of a large number of books originally published in alien countries. The latest edition available in the United States before the war is being republished in each case.

A publisher obtaining a license for a particular book in the Custodian's list agrees to reproduce all or a substantial part of the work by photo-offset or otherwise in the original language of publication. Licenses are issued upon a competitive basis for a period of five years; they are non-exclusive, but the Custodian agrees not to grant a second license pertaining to any work for a period of six months after the date of the first license. A royalty fee of 15% is charged upon the list or retail price of any book reproduced; a publisher, however, is permitted the recoupment of the cost of publication before royalties become payable.

If a publisher suggests a title not included in the Custodian's list, the publisher suggesting the title for republication may submit a request for a license under essentially the same terms and conditions as given above. A license may then be granted at the discretion of the Custodian without exploratory work being done by the Custodian or other bids being asked.

Translations are licensed for the life of the copyright. In this case, the recoupment of the original publication, translation, and exploitation costs is allowed before any royalty will be charged. Thereafter, royalty at the rate of 10% of the list or retail sale price will be payable to the Custodian. Otherwise, licenses to translate and revise works are given under substantially the same terms as those indicated for republication in the original language. The translator and publisher will hold the copyright for the specific translation.

Among the books already announced as licensed for reproduction are many which are of interest to mathematicians. A selected list of these titles is given below. All communications with respect to the purchase of these works should be addressed to the licensee designated for each title, *not* to the Office of Alien Property Custodian.

Biezeno, Cornelius B. *Technische Dynamik*. Berlin, J. Springer, 1939. Licensee: Edwards Brothers, Ann Arbor; $27.80. (Original price: $31.20).

Bjerknes, V. F. K., and others. *Physikalische Hydrodynamik, mit Anwendung auf die dynamische Meteorologie*. Berlin, J. Springer, 1933. Licensee: Edwards Brothers, Ann Arbor; $18.00. (Original price: $27.50).

Born, Max. *Optik. Ein Lehrbuch der elektromagnetischen Lichttheorie*. Berlin, J. Springer, 1933. Licensee: Edwards Brothers, Ann Arbor; $13.00. (Original price: $14.20).

Courant, Richard, and Hilbert, David. *Methoden der mathematischen Physik*. Berlin, J. Springer. v. 1, 2 Aufl., 1931; v. 2, 1937. Licensee: Interscience Publishers, New York; $14.00, 2 vol. in 1. (Original price: v. 1, $12.32; v. 2, $15.92).

Cranz, Carl Julius. *Lehrbuch der Ballistik*. Berlin, J. Springer, 1925–1936. Licensee: Edwards Brothers, Ann Arbor. Complete work, $55.00; v. 1, $24.00; v. 2, $16.50; v. 3, $15.50; v. 4, $11.00. (Original price: $63.00).

Ertel, Hans. *Methoden und Probleme der dynamischen Meteorologie*. Berlin, J. Springer, 1938. Licensee: Edwards Brothers, Ann Arbor, $4.50. (Original price: $5.50).

Flügge, Wilhelm. *Statik und Dynamik der Schalen*. Berlin, J. Springer, 1934. Licensee: Edwards Brothers, Ann Arbor, $5.75. (Original price: $9.00).

Frank, Philipp, and Mises, R. Von. *Die Differential und Integralgleichungen der Mechanik und*

Physik. 2 Aufl. Braunschweig, F. Vieweg und Sohn, 1930–1935. Licensee: M. S. Rosenberg, New York, $27.50. (Original price: v. 1, $22.40; v. 2, $26.00).

Hilbert, David. *Grundlagen der Mathematik*. Berlin, J. Springer, 1934–1939. Licensee: Edwards Brothers, Ann Arbor. Complete work, $25.00; v. 1, $12.40; v. 2, $13.20. (Original price: $32.65).

Jahnke, Eugen, and Emde, Fritz. *Funktionentafeln mit Formeln und Kurven*. 3 Aufl. Leipzig und Berlin, B. G. Teubner, 1938. Licensee: G. E. Stechert, New York; $3.00. (Original price: $6.00). Second licensee: Dover Publications, New York; $3.50. (Note: The Dover edition also contains a 76 page addendum, "Index of tables of the elementary transcendentals" from the 1933 edition).

Jordan, Pascual. *Einführung in die Gedankenwelt der modernen Physik*. Braunschweig, F. Vieweg und Sohn, 1938. Licensee: Philosophical Library, New York; English translation, $4.00. (Original price: $2.40).

Peters, Jean. *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Grades*. Leipzig and Berlin, B. G. Teubner, 1938. Licensee: D. Van Nostrand, New York; $7.50. (Original price: $10.00).

Prandtl, Ludwig. *Vier Abhandlungen zur Hydrodynamik und Aerodynamik*, Göttingen, Kaiser Wilhelm Institute, 1927. Licensee: Edwards Brothers, Ann Arbor; $3.50. (Original price: $1.60).

Stumpff, Karl. *Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung*. Berlin, J. Springer, 1939. Licensee: Edwards Brothers; $5.25. (Original price: $14.65).

Waerden, Bartel Leendert Van Der. *Moderne Algebra*. Berlin, J. Springer, v. 1, 2 Aufl., 1937, v. 2, 1931. Licensee: Frederich Ungar, New York; $6.00. (Original price: $12.80).

MATHEMATICIANS AND THE NEW SELECTIVE SERVICE REGULATIONS

On January 10, 1944, Selective Service Headquarters released amended Local Board Memoranda 115 and 115B, which become effective on February 1, and also an amended Activity and Occupational Bulletin No. 33-6, which becomes effective on February 15. These vitally concern all chairmen of departments of mathematics, for they deal with the occupational classification of both teachers and undergraduate students. This communication contains a summary of the provisions of these directives.

Mathematicians are still included in the List of Critical Occupations. It is specifically stated that the titles included in Part II (Professional and Scientific Occupations) of the List of Critical Occupations "shall be considered as also including persons engaged in full-time teaching of these professions. A person may be considered as engaged in full-time teaching if he devotes not less than 15 hours per week in contact with students in actual classroom or laboratory instruction." The problem of securing the proper classification of a *teacher* of mathematics differs in accordance with the age group into which he falls.

I. Deferment of Mathematicians in 18–21 Age Group

All requests for new or additional occupational deferments for registrants in this age group who are teachers of mathematics must be made on Director of Selective Service Form 42-A Special. An original and two copies of this form must be presented by the employer to the State Director in whose state is located the registrant's principal place of employment.

Effective February 1, no registrant in this group, at the time he is classified, may be considered as a "necessary man" entitled to be placed in Class II-A or Class II-B unless:

(a) the State Director of Selective Service has endorsed his Form 42-A Special with a statement that, based on the information furnished therein, he recommends that the local board except the registrant from the general restrictions against occupational deferment of registrants aged 18 through 21.

(b) he is classified as (1) belonging to the Personnel of the Merchant Marine or Army Transportation Corps or (2) as a student who qualifies under AOB 33-6.

Registrants in this group who are classified II-A or II-B on February 1, 1944, will, in general, not have deferments terminated before the expiration date. (This, however, does not apply to students covered in the new AOB 33-6.)

The President and Secretary of the Society have addressed a memorandum to State Directors of Selective Service, calling attention to the vital need for these mathematicians to handle the mathematics in the Army Specialized Training Program and the Navy College Training Program.

II. Deferment of Mathematicians in 22–37 Age Group

In making requests for occupational deferment, one copy of Form 42-A must be filed with the local board, as heretofore. The local board may, without going further, classify the registrant in II-A or II-B, as requested.

If the registrant is not classified in II-A or II-B and if reference to the United States Employment Service was not made by the local board prior to classification in I-A, I-A-O or IV-E, it must be made immediately after such classification. Upon suitable certification from the USES, the local board must reopen the case. The cases of registrants who are qualified for professional and scientific occupations will be forwarded by the USES to the National Roster. In all cases in which reference to the USES is made, the local board is directed not to issue an order to report for induction until it has received a report from the USES or until the expiration of 30 days after referral, whichever occurs first. If, during the 30-day period, the USES certifies to the local board that the registrant is qualified and that his removal would adversely affect the maintenance of his employer's required production, the local board must reopen the case and consider the new evidence as a basis for further deferment.

Even though a registrant is not employed in the area in which his local board is located, the registrant's local board must refer the case to the local USES office in the local board area and any further reference of the case to the USES office in which the registrant is employed will be undertaken and accomplished by the USES.

In view of these provisions, *department chairmen are urged to register their needs for personnel with the USES office where the college or university is located and in the USES office where the registrant's local board is located.* It is probable that the USES will be unable to supply any mathematicians but this is the best way to establish the fact that there is a shortage. If the USES does not have

calls for mathematicians, physicists, engineers, chemists, et cetera, it will have no reason to believe that there is a shortage.

The USES offices are required to refer cases involving scientific personnel to the National Roster. (Information has come to the Secretary to the effect that the USES offices are not now referring cases to the Roster.) If department chairmen believe that the advice given by the local USES office to the local board is inadequate or incorrect, the employer is advised to call the attention of the USES office to the instructions in Sections 5300–5309 of the USES Manual (which contain instructions involving the National Roster). If Roster advice is not sought, the chairman will do well to report all facts in the case directly to the Roster.

Department chairmen are urged to make the original requests for occupational deferment as strong as possible and to make all possible appeals. Despite the curtailment of certain college training programs, chairmen are advised to continue requests for deferment, for the shortage of qualified mathematicians is still critical.

III. Deferment of Undergraduates

In Activity and Occupation Bulletin 33-6 the position is taken that the Army and Navy Training Programs will provide adequately for the needs of the armed forces. Student deferment is for civilian needs in war production and in support of the war effort.

Students who will graduate on or before July 1, 1944, in a considerable number of fields (including engineering branches, bacteriology, chemistry, forestry, geophysics, mathematics, meteorology, optometry, pharmacy, and physics including astronomy) are eligible for occupational deferment on certification by the institution and the National Roster.

An undergraduate who is a full-time student of chemistry, engineering, geology, geophysics, or physics who will graduate after July 1, 1944, will be eligible for occupational deferment, if he is properly certified by his institution and the National Roster and will graduate within 24 months from the date of certification and provided the national quota of 10,000 is not exceeded. Mathematics was not included in the list mentioned above, in spite of the strong protests of our Washington representatives. For undergraduate students in the 18–21 age group, requests for deferment are made in duplicate on DSS Form 42 Special; for students over 22 years of age, Form 42 is used.

A national quota of 10,000 has been established for students who should be occupationally deferred at any one time by reason of pursuing courses of study in chemistry, engineering, geology, geophysics, and physics. Students deferred to graduate on or before July 1, 1944, and students deferred for reasons other than pursuing a course of study will not be counted against this quota.

Requests for information needed by the National Roster to set up institutional quotas, et cetera, have already gone out to college presidents. Information regarding the procedures to be followed by institutions and departments con-

cerned will be issued by the National Roster to college presidents in the near future. The departments concerned should be on the lookout for such information.

Department chairmen are requested to inform the Secretary of cases in which mathematicians are improperly classified. In this manner the War Policy Committee, through its Washington representatives, will be better able to present the needs of the mathematicians.

January 17, 1944.

J. R. KLINE,
Secretary, War Policy Committee

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-SEVENTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-seventh annual meeting of the Mathematical Association of America was held at Chicago, Illinois, on Saturday and Sunday, November 27 and 28, 1943, in conjunction with meetings of the American Mathematical Society. About two hundred and seven persons attended the meetings, including the following one hundred and thirty-six members of the Association:

R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
C. B. ALLENDOERFER, Haverford College
MAX ASTRACHAN, Antioch College
H. G. AYRE, Western Illinois State Teachers College
W. L. AYRES, Purdue University

R. W. BABCOCK, Kansas State College
RUTH MASON BALLARD, Wright Junior College
HARRY BATEMAN, California Institute of Technology
S. F. BIBB, Illinois Institute of Technology
B. H. BISSINGER, Michigan State College
L. M. BLUMENTHAL, University of Missouri
H. C. BOARDMAN, Chicago Bridge and Iron Company
J. M. BOOKSTEIN, Army Air Forces
FANNIE W. BOYCE, Wheaton College
ANGELINE J. BRANDT, Wheaton College
R. W. BRINK, University of Minnesota
E. L. BUELL, Northwestern University
H. E. BURNS, Indiana University Extension
HERBERT BUSEMANN, Illinois Institute of Technology

W. D. CAIRNS, University of New Mexico
W. B. CARVER, Cornell University
W. B. CATON, Illinois Institute of Technology
W. F. CHENEY, JR., University of Connecticut
E. W. CHITTENDEN, University of Iowa
LAURA E. CHRISTMAN, Senn High School, Chicago
R. V. CHURCHILL, University of Michigan
A. B. COBLE, University of Illinois
L. W. COHEN, University of Kentucky
N. B. CONKWRIGHT, University of Iowa
A. H. COPELAND, University of Michigan
J. J. CORLISS, De Paul University
D. R. CURTISS, Northwestern University

JOHN DECICCO, Illinois Institute of Technology
JESSE DOUGLAS, Brooklyn College
ARNOLD DRESDEN, Swarthmore College
D. M. DRIBIN, Signal Corps

J. P. ESPOSITO, Crane Technical High School, Chicago
H. J. ETTLINGER, University of Texas
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

G. M. EWING, University of Missouri

J. V. FINCH, University of Chicago
N. J. FINE, Purdue University
EDWARD FLEISHER, Brooklyn College
L. R. FORD, Illinois Institute of Technology
J. S. FRAME, Michigan State College
T. C. FRY, Bell Telephone Laboratories

J. S. GEORGES, Wright Junior College
J. W. GIVENS, JR., Northwestern University
MICHAEL GOLOMB, Purdue University
CORNELIUS GOUWENS, Iowa State College
L. M. GRAVES, University of Chicago
V. G. GROVE, Michigan State College

D. W. HALL, University of Maryland
P. R. HALMOS, Syracuse University
R. W. HAMMING, University of Illinois
W. L. HART, University of Minnesota
M. H. HEINS, Illinois Institute of Technology
E. D. HELLINGER, Northwestern University
EDWARD HELLY, Illinois Institute of Technology
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
A. S. HOUSEHOLDER, University of Chicago
H. K. HUGHES, Purdue University
RALPH HULL, University of Nebraska
M. GWENETH HUMPHREYS, Sophie Newcomb College
MILDRED HUNT, Illinois Wesleyan University

M. H. INGRAHAM, University of Wisconsin

B. W. JONES, Cornell University

DORA E. KEARNEY, State Teachers College, Cedar Falls, Iowa
M. W. KELLER, Purdue University
J. R. KLINE, University of Pennsylvania
L. A. KNOWLER, University of Iowa
W. C. KRATHWOHL, Illinois Institute of Technology

E. P. LANE, University of Chicago
LUISE LANGE, Woodrow Wilson Junior College
R. E. LANGER, University of Wisconsin
C. G. LATIMER, University of Kentucky
D. H. LEAVENS, University of Chicago
MAYME I. LOGSDON, University of Chicago
A. T. LONSETH, Iowa State College

JANET MACDONALD, Hinds Junior College

C. C. MACDUFFEE, University of Wisconsin
MORRIS MARDEN, University of Wisconsin
W. T. MARTIN, Syracuse University
J. N. MCCLELLAND, Drake University
GLADYS B. MCCOLGIN, Indianapolis, Ind.
KARL MENGER, University of Notre Dame
E. J. MICKLE, Ohio State University
H. J. MISER, Illinois Institute of Technology
C. W. MORAN, Lane Technical School, Chicago
MARSTON MORSE, Institute for Advanced Study
E. J. MOULTON, Northwestern University

A. L. NELSON, Wayne University
C. V. NEWSOM, University of New Mexico
IVAN NIVEN, Purdue University
E. A. NORDHAUS, Michigan State College
E. P. NORTHROP, University of Chicago

ISAAC OPATOWSKI, Illinois Institute of Technology

W. V. PARKER, Louisiana State University
B. C. PATTERSON, Hamilton College

TIBOR RADÓ, Ohio State University
G. T. RAINICH, University of Michigan
RUTH B. RASMUSEN, Woodrow Wilson Junior College
C. B. READ, University of Wichita
W. T. REID, University of Chicago
HAIM REINGOLD, Illinois Institute of Technology
W. H. ROEVER, Washington University

R. G. SANGER, University of Chicago
E. W. SCHREIBER, Western Illinois State Teachers College
A. A. SCHWARTZ, Textile Evening High School, New York, N. Y.
M. E. SHANKS, University of Missouri
JACK SILBER, Illinois Institute of Technology
H. A. SIMMONS, Northwestern University
G. W. SMITH, University of Kansas
I. S. SOKOLNIKOFF, University of Wisconsin
C. F. STEPHENS, Great Lakes Naval Training Station
M. H. STONE, Harvard University
E. B. STOUFFER, University of Kansas
J. L. SYNGE, Ohio State University

R. M. THRALL, University of Michigan

W. R. UTZ, University of Notre Dame

H. S. WALL, Northwestern University
WARREN WEAVER, Rockefeller Foundation
K. W. WEGNER, Carleton College
J. V. WEHAUSEN, University of Missouri
E. T. WELMERS, Michigan State College
M. E. WESCOTT, Northwestern University
G. T. WHYBURN, University of Virginia
L. R. WILCOX, Illinois Institute of Technology
R. L. WILDER, University of Michigan
K. P. WILLIAMS, Indiana University
A. H. WILSON, Haverford College
R. S. WOLFE, Northwestern University
J. H. ZANT, Oklahoma A. and M. College

The meetings were held in the auditorium of the Museum of Science and Industry in Jackson Park at 57th Street and South Shore Drive, and the hotel headquarters were at the Hotels Windermere. Lunches were available at a convenient cafeteria in the Museum.

Sessions of the American Mathematical Society were held on Friday and on Saturday morning and afternoon. On Friday at 7:45 p.m. the seventeenth Josiah Willard Gibbs Lecture was given by Professor Harry Bateman on the topic, "The control of elastic fluids." Two addresses were given by invitation, one by Professor Reinhold Baer, at 11:30 a.m. on Friday, on "The higher commutator subgroups," and the other by Professor Marston Morse, at 2:00 p.m. Saturday, on "New settings for topology in analysis."

The Mathematical Association held sessions Saturday evening and Sunday morning, the program having been prepared by a committee consisting of A. H. Copeland, L. R. Ford, and W. T. Reid, *Chairman*. The program follows.

FIRST SESSION OF THE ASSOCIATION

1. "College mathematics during reconstruction," retiring presidential address by Professor R. W. Brink, University of Minnesota.

2. "Focal properties of optical and electromagnetic systems of revolution" by Professor J. L. Synge, Ohio State University.

SECOND SESSION OF THE ASSOCIATION

3. "Mathematics textbooks and the introductory courses in college mathematics" by Professor H. P. Evans, University of Wisconsin.

4. "The nature of mathematical proof" by Professor R. L. Wilder, University of Michigan.

5. "Mathematical biophysics and the central nervous system" by Professor A. S. Householder, University of Chicago.

MEETINGS OF THE BOARD OF GOVERNORS

The Board met Saturday afternoon at 3:30 and for a few minutes at noon on Sunday.

The following seventeen persons were elected to membership on applications duly certified:

R. F. BELDING, A.B.(Amherst) Instr., Vermont Acad., Saxtons River, Vt.

RUTH A. BRENDEL, A.B.(Buffalo) Instr., Univ. of Buffalo, Buffalo, N. Y.

SAMUEL BROYDE, J.D.(De Paul) Instr., Northwestern Univ., Evanston, Ill.

J. W. CRISPIN, JR. Student, Univ. of Michigan, Ann Arbor, Mich.

J. M. DANSKIN, JR., A.B.(U.C.L.A.) Seaman, Second Class, U.S.N.R., Wright Jr. Coll., Chicago 34, Ill.

E. S. J. GOUGH, B.A., B.Ed.(Montreal) Prof., Math. and Sci., Jacques Cartier Normal School, Montreal, P. Q., Canada

A. E. HALLERBERG, A.M.(Illinois) Instr., Illinois Coll., Jacksonville, Ill.

MARY A. LEE, A.M.(Wisconsin) Grad. Asst., Cornell Univ., Ithaca, N. Y.

C. E. MILLER, Ph.D.(Toronto) Prof., Univ. of New Brunswick, Fredericton, N. B., Canada

AGNES L. MORGAN, A.M.(Columbia) Teacher, Brackenridge High School, San Antonio, Tex.

IRMA R. MOSES, A.B.(Cornell) Grad. Asst., Cornell Univ., Ithaca, N. Y.

JAMES PALLADINO, B.S.(Long Island Univ.) Instr., Physics, Long Island Univ., Brooklyn, N. Y.

J. S. ROSEN, Ph.D.(Washington Univ.) Asso. Prof., Eastern New Mexico Coll., Portales, N. M.

F. E. ROTHCHILD, M.S.(Louisiana State) Prof., Math. and Phys. Educ., Arkansas A. and M. Coll., Monticello, Ark.

C. T. RUDDICK, Ph.D.(Pennsylvania) Asst. Prof., Mount Union Coll., Alliance, Ohio

BENJAMIN SLEPIN, LL.B.(South Jersey Law Sch.) Instr., Frankford Arsenal Trade School, Philadelphia, Pa.

ADRIAN STRUYK, M.E.(Stevens Inst. of Tech.), A.M.(Columbia) Teacher, High School, Clifton, N. J.

The Secretary reported the deaths of the following members during the year 1943:

W. M. Carruth, Professor of mathematics, Hamilton College. (January 23, 1943)

C. H. Currier, Associate Professor emeritus of mathematics, Brown University. (January 5, 1943)

H. H. Dalaker, Professor emeritus of mathematics, University of Minnesota. (May 20, 1943)

E. L. Dodd, Professor of actuarial mathematics, University of Texas. (January 9, 1943)

Guido Fubini, Institute for Advanced Study. (June 6, 1943)

Lillian Hackney, Professor emeritus of mathematics, Marshall College. (February 4, 1943)

G. A. Harter, Professor emeritus of mathematics, University of Delaware. (July 22, 1943)

H. E. Hawkes, Dean, Columbia University. (May 4, 1943)

E. R. Hedrick, recently Vice President and Provost, University of California at Los Angeles. (February 3, 1943)

L. L. Locke, Brooklyn. (August 28, 1943)

R. L. Menuet, Professor of mathematics, Tulane University. (May 9, 1943)

H. H. Mitchell, Professor of mathematics, University of Pennsylvania. (March 13, 1943)

A. G. Rau, Dean, Moravian College. (1942)

N. S. Risley, Assistant Professor of mathematics, Fenn College. (December 30, 1942)

H. M. Showman, Lecturer and Registrar, University of California at Los Angeles. (June 24, 1943)

Clara E. Smith, Professor emeritus of mathematics, Wellesley College. (May 12, 1943)

E. B. Van Vleck, Professor emeritus of mathematics, University of Wisconsin. (June 2, 1943)

J. J. Westemeier, Professor of mathematics, Dowling College. (February 1, 1943)

J. E. Williams, Dean and Professor of mathematics, Virginia Polytechnic Institute. (April 19, 1943)

The Secretary-Treasurer announced that he would prepare a complete financial report for the year after January 1, 1944, and that this report would be published in the MONTHLY. He reported that the financial operations for the

year would apparently compare favorably with the budget items as estimated at the beginning of the year.

Letters were read from several members protesting against the holding of sessions of the Association on Sunday. No action was taken on the question; but in the discussion the opinion was expressed that such Sunday sessions were necessitated by war conditions, and that there was no intention to continue the practice after the present emergency.

It was voted, on the request of the Secretary-Treasurer, to have a thorough audit of the accounts of the Association at the end of the year, the Finance Committee to arrange for such an audit.

H. M. Gehman of the University of Buffalo was elected a member of the Finance Committee for a term of four years to succeed R. E. Langer.

The Editor-in-Chief nominated and the Board elected the following associate editors of the MONTHLY for the year 1944:

E. F. BECKENBACH	H. P. EVANS	M. R. HESTENES
L. M. BLUMENTHAL	B. F. FINKEL	B. W. JONES
N. B. CONKWRIGHT	J. S. FRAME	R. E. LANGER
H. S. M. COXETER	ORRIN FRINK, JR.	C. V. NEWSOM
W. M. DAVIS	MARJORIE GROVES	VIRGIL SNYDER
OTTO DUNKEL		MARIE J. WEISS

At the meeting on Sunday, R. E. Langer was nominated to succeed T. C. Fry as the representative of the Association on the National Research Council for a term of three years beginning July 1, 1944.

ANNUAL BUSINESS MEETING

The annual business meeting of the Association was held on Sunday morning at 9:30, President Cairns presiding.

The results of the election of officers were announced as follows:

First Vice-President for the term 1944–45: W. M. Whyburn, University of California at Los Angeles.

Governors at Large for the term 1944–46: R. G. Sanger, University of Chicago, and Morgan Ward, California Institute of Technology.

W. B. CARVER, Secretary-Treasurer

THE FALL MEETING OF THE INDIANA SECTION

The twenty-first annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Friday, October 29, 1943, in conjunction with the fall meeting of the Indiana Academy of Science. Professor P. M. Pepper, Chairman of the Indiana Section of the Association, presided at the morning session, and Professor W. E. Edington, Chairman of the Mathematics Section of the Academy, presided at the afternoon session.

Forty-three individuals registered at the meetings, including the following

twenty-five members of the Association: Emil Artin, W. L. Ayres, K. W. Crain, J. E. Dotterer, W. E. Edington, P. D. Edwards, Michael Golomb, G. H. Graves, Cora B. Hennel, M. W. Keller, E. L. Klinger, Gladys B. McColgin, H. A. Meyer, G. T. Miller, Paul Muehlman, P. M. Pepper, J. C. Polley, Maxwell Reade, D. A. Rothrock, C. P. Sousley, M. S. Webster, J. W. Wiley, K. P. Williams, H. E. Wolfe, A. J. Zanolar.

At the business meeting the following officers were elected for next year: Chairman, Emil Artin, Indiana University; Vice-Chairman, W. L. Ayres, Purdue University; Secretary, M. W. Keller, Purdue University. It was decided to hold the next meeting in conjunction with the annual meeting of the Indiana Academy of Science.

The following papers were presented:

1. *Isohedral polyhedra*, by Leon Alaoglu and Dr. J. H. Giese, Purdue University, introduced by Professor W. L. Ayres.

In this paper isohedral and isogonal polyhedra were defined, and it was indicated that the classical regular polyhedra are both isohedral and isogonal. Attention was then directed to polyhedra which are isohedral, but which have isogonality replaced by the weaker property that equal numbers of faces meet at the vertices. It was pointed out that for finite polyhedra of genus zero (topological spheres) the Euler polyhedron formula reduces the possibilities to the usual five, ranging from tetrahedron to icosahedron. Constructions using the greatest possible numbers of unequal edges per face were devised to show the existence of all of these five types except the icosahedron with scalene triangular faces. In the case of genus one (topological tori) the Euler formula reduces the possibilities to triangular, quadrilateral, and hexagonal faces. Constructions were devised to establish the existence of isohedral tori with $12n$ ($n \geqq 3$) triangular faces, and of isohedral tori with $8n$ ($n \geqq 4$) quadrilateral faces.

2. *The elementary functions*, by Professor Emil Artin, Indiana University.

Professor Artin showed how to introduce the elementary functions e^x, $\log x$, $\cos x$ and $\sin x$ in a completely rigorous manner by using only the simplest facts relating to limits. The results thus obtained cover all properties of these functions, including the infinite product for $\sin x$. It is thus possible to have all these functions available from the beginning in a course in advanced calculus.

3. *A method for the solution of algebraic or transcendental equations*, by Dr Michael Golomb, Purdue University.

The speaker pointed out that the familiar methods for the solution of equations have certain shortcomings. (Newton's and Horner's methods apply only to real roots, while Graeffe's method applies only to algebraic equations, *etc.*) He derived a new method based upon Hadamard's investigations of the singularities of functions defined by Taylor series. The symmetric functions of the zeros of smallest absolute value were given as limits of quotients of persymmetric determinants involving successive coefficients in the Maclaurin expansion of the reciprocal of the function.

4. *Some developments in the analytic theory of continued fractions*, by Dr. Marion Wetzel, Indiana University, introduced by Professor K. P. Williams.

This address dealt with certain recent contributions to the analytic theory of continued fractions. These contributions have attempted to bring together many isolated results, and fit them into a larger analytic structure. The speaker regarded the continued fraction as an infinite sequence of linear fractional transformations in the complex plane. The class of continued fractions

$$\frac{1}{b_1+z}-\frac{a_1^2}{b_2+z}-\frac{a_2^2}{b_3+z}-\cdots,\ (a_p\neq 0)$$

for which the quadratic form

$$\sum_{p=1}^{n}\Im(b_p+Z)X_p^2-2\sum_{p=1}^{n-1}\Im(a_p)X_pX_{p-1}$$

is positive definite for all values of $\Im(z)>0$ was discussed. These continued fractions include the classical case in which $\Im(b_p)=\Im(a_p)=0$, and also the case $\Im(b_p)\geqq 0$, $\Im(a_p)=0$, discussed in a paper by Hellinger and Wall in the *Annals of Mathematics*, vol. 44, 1943, pp. 103–127. Necessary and sufficient conditions for positive definiteness in terms of the imaginary parts of the coefficients in the quadratic form were given. The speaker cited some applications of this characterization, including connections with theorems on convergence regions.

5. *Remarks on surfaces*, by Professor J. W. T. Youngs, Purdue University, introduced by Professor M. W. Keller.

Professor Youngs made some expository comments on classical and modern surface theory.

The afternoon session was devoted to a panel discussion of the mathematics training offered for the armed forces in Indiana colleges. The following topics were brought before the meeting:

I. *The content of A. S. T. P. courses with special attention to the relegation of theory to a place of minor importance*, by Professor W. L. Ayres, Purdue University.

II. *Standards for the retention of men in the A. S. T. P.*, by Professor K. P. Williams, Indiana University.

III. *Content of the V-12 program (excluding navigation)*, by Professor J. C. Polley, Wabash College.

IV. *Navigation in the V-12 program*, by Professor R. F. McDaid, Indiana State Teachers College.

V. *Content of courses in the advanced navy program in light of the preparation of the students enrolled*, by Professor Paul Pepper, University of Notre Dame.

VI. *Army pre-flight courses*, by Professor J. L. Beal, Butler University.

VII. *Navy pre-flight courses*, by Professor W. E. Edington, DePauw University.

VIII. *Navy primary and secondary flight courses*, by Professor P. D. Edwards, Ball State Teachers College.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS, Normal, Ill., May 12–13, 1944
INDIANA, Indianapolis, November 10, 1944
IOWA, Cedar Rapids, April 15, 1944
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN, Ann Arbor, March 18, 1944
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1944
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

For Army, Navy and regular College Courses

CURTISS & MOULTON'S

Essentials of Trigonometry with Applications

A relatively brief treatment of the essential principles of plane and spherical trigonometry. Applications include problems connected with maps and with artillery fire, where the mil and other terms are explained and problems of aiming briefly treated; and problems in navigation. Abundant exercises. *With Tables, 276 pages, $2.25. Without Tables, 182 pages, $2.00. Tables separately, $1.25*

Essentials of Spherical Trigonometry with Applications

Contains the chapter on spherical trigonometry and the section on applications from the complete book. *Paper, 63 pages, $.60*

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

Raymond W. Brink's

INTERMEDIATE ALGEBRA

THE purpose of this text is to supply the needs of students who have had only one year of high school algebra and who desire to prepare for strictly college mathematics or for work in statistics, physics, and other sciences. The book gives thorough preparation for college mathematics. While helping the student to acquire good skill in technical manipulation it stresses logical development and exact statement. It gives a large number of examples solved in the text and provides an abundance and variety of well-graded exercises. $1.50

D. APPLETON-CENTURY COMPANY

35 West 32nd Street, New York 1, N.Y.

TRIG PROBLEMS ARE EASY...

with these new Slide Rules

The improved arrangement of the scales on these new Log Log Duplex Trig* and Log Log Duplex* Decitrig Slide Rules greatly simplifies the handling of trig problems. The trig scales refer directly to the CD scales so that trigonometrical functions can be used as factors without having their numerical values determined.

*Trade Mark

KEUFFEL & ESSER CO.

NEW YORK · HOBOKEN, N. J.

Chicago St. Louis San Francisco Los Angeles Detroit Montreal

Outline of the History of Mathematics

by RAYMOND CLARE ARCHIBALD

Fifth edition, June 1941, ii, 76 pages

THIS thoroughly revised and considerably enlarged edition is published by the Mathematical Association to meet a constant demand for up-to-date information on the History of Mathematics, not available in any other single English work. The syllabus, twenty pages of references to other material and sources, and an entirely new Index of Names, provide an excellent basis for a teacher to conduct a semester or year course in this field, or for a student wishing greatly to extend his knowledge. Earlier editions of the Outline have been reviewed very favorably throughout the world. Our expectation is that this fifth edition published within ten years, and more fundamentally revised and extended than any other edition, will continue to meet a need in this country and elsewhere.

Price 75 cents a copy, postpaid, remittance with order

No discount in price to anyone

MATHEMATICAL ASSOCIATION OF AMERICA
McGRAW HALL, CORNELL UNIVERSITY
ITHACA, NEW YORK

ANNUAL REPORT

Outstanding New Texts for War Training Courses

Aircraft Analytic Geometry

By J. J. APALATEGUI, Douglas Aircraft Co., Inc., and L. J. Adams, Santa Monica Junior College. 286 pages, $3.00.

Deals with the application of the methods of plane and solid analytic geometry to the solution of certain problems that arise in the design, lofting, tooling, and engineering of airplanes.

Mathematics for Navigators

By Comdr. DELWYN HYATT, USN (Ret.); U. S. Merchant Marine Academy, Kings Point, N.Y., and Comdr. B. M. Dodson, USNR; U. S. Merchant Marine Cadet Basic School, Pass Christian, Miss. 110 pages, $1.25.

A refresher in mathematics for those who are about to study navigation, with particular reference to the probems involved in this field.

Military Applications of Mathematics

By PAUL P. HANSON, Manlius School, N.Y. 425 pages, textbook edition, $2.40.

Brings together in one volume the problems in all branches of the Armed Forces that can be solved with a background of high school mathematics.

Plane and Spherical Trigonometry. ***Alternate Edition***

By LYMAN M. KELLS, WILLIS F. KERN and JAMES R. BLAND, U. S. Naval Academy. 371 pages, $2.00. With tables, $2.75.

An edition of the authors' well known *Plane and Spherical Trigonometry,* adapted especially to conditions of war. Contains many military and nautical applications.

Mathematics Essential to Electricity and Radio

By NELSON M. COOKE, Lieut., USN. Executive Officer, Radio Materiel School, Naval Research Laboratory, Washington, D.C., and JOSEPH B. ORLEANS, George Washington High School, New York, N.Y. 418 pages, textbook edition, $2.40.

Gives the student a quick working knowledge of the essentials of mathematics required for electricity and radio.

Mathematics of Flight

By JAMES NAIDICH, Manhattan High School of Aviation Trades, New York, N.Y. 415 pages, textbook edition, $2.20.

Includes those topics on algebra, geometry, and trigonometry needed by the student of aeronautics.

Elementary Mathematics for the Machine Trades

By JOHN J. WEIR, Sperry Gyroscope Co. 193 pages, textbook edition, $1.60.

Presents the essential mathematics required for machine shop work and for the solution of everyday problems of the machinist.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

Especially designed to meet the basic mathematical requirements of the Armed Services and the essential industries

Basic Mathematics for War & Industry

By DAUS, GLEASON & WHYBURN

This new book provides text material in arithmetic, algebra, plane and solid geometry, and plane and spherical trigonometry (including elements of navigation). All mathematical principles are carefully stated and thoroughly illustrated with practical and up-to-date examples and problems chosen from a variety of fields. All the topics needed for courses in Shop Work are covered, as well as those basic to flight mathematics, to the training given to artillery officers including anti-aircraft, and to other industrial and military training. It is written by professional mathematicians who have had wide experience in designing and teaching courses of this type. Paul H. Daus and William M. Whyburn are both on the staff of the University of California at Los Angeles. John M. Gleason is on the staff of San Diego State College.

Published February 1st. *277 pages.* *Illus.* *$2.00*

The Macmillan Company · 60 Fifth Avenue · New York 11

BATTLE

ש

TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

APR 12 1944

VOLUME 51 NUMBER 3

CONTENTS

Types of Symmetries . . . . . T. L. WADE, R. H. BRUCK 123
Boolean Representation Theory . . . . . E. R. STABLER 129
An Electrical Chinese Ring Puzzle . . . . . A. H. BEILER 133
Skew-Symmetric Matrices and Projective Geometry . . . . . H. SCHWERDTFEGER 137
Mathematics, 200 B.C.–600 A.D. . . . . MAX DEHN 149
Clubs and Allied Activities 157
Discussions and Notes . . . F. L. DENNIS, RICHARD BELLMAN 159
Problems and Solutions 162
News and Notices 172
War Information 173
The Republication of Foreign Mathematical Tables 173
Quotas in Military Training Programs 174
Central Clearing Agency for Accreditation 175
Enlisted Men Entering the Navy V-12 Program 175
The Navy V-5 Program 176
The Mathematical Association of America 177
Report of the Treasurer for the year 1943 177
Kansas Section Meeting 181
Philadelphia Section Meeting 182
Calendar of Future Meetings 184

MARCH 1944